**Functional Time Series**

- Functional data objects collected sequentially over time are characterized as functional time series.
- Each term in series is a function \( X_i(t) \) defined for \( t \) taking values in some interval \([a, b] \).
- Long continuous records of temporal sequence segmented into curves over consecutive time intervals.
- Examples: daily price curves of financial transactions, daily patterns of environmental data.

**The FAR(1) Model**

Let \( \{X_i\} \) be a stationary and \( \alpha \)-mixing functional sequence in some separable Hilbert space \( \mathbb{H} \) with the usual definition of \( \alpha \)-mixing coefficients introduced by Rosenblatt (1956). It is endowed with inner product \((\cdot, \cdot)\) and corresponding norm \( \| \cdot \| \), and with orthonormal basis \( \{e_j : j = 1, \cdots, \infty\} \). We consider the following first-order nonparametric functional autoregression model, namely FAR(1):

\[
X_{i+1} = \Psi(X_i) + \varepsilon_{i+1}, \quad i = 1, 2, \ldots,
\]

- \( \Psi \) is the autoregressive operator mapping functions from \( \mathbb{H} \) to \( \mathbb{H} \).
- Innovations \( \varepsilon_i \)'s are i.i.d. \( \mathbb{H} \)-valued r.v.'s with \( E(\varepsilon_i | X_i) = 0 \) and \( E(|\varepsilon_i|^2) = \sigma_i^2(\varepsilon_i < \infty) \).

**Estimation of \( \Psi \)**

Estimation of \( \Psi \) is given by the functional version of Nadaraya-Watson estimator of time series

\[
\hat{\Psi}_h(\chi) = \frac{\sum_{i=1}^{n-1} X_i K(h^{-1}d(\chi, X_i))}{\sum_{i=1}^{n-1} K(h^{-1}d(\chi, X_i))},
\]

where \( \chi \) is a fixed element in \( \mathbb{H} \), \( K \) is a kernel function, \( d(\cdot, \cdot) \) is a semi-metric defined to measure the proximity between the two elements in \( \mathbb{H} \), and \( h \) is a bandwidth sequence, tending to zero as \( n \) tends to infinity.

Some assumptions are made on the kernel:

- \( K(\cdot) \) is supported on \([0, 1]\), has a continuous derivative on \([0, 1]\);
- \( K'(\cdot) \leq 0 \) and \( K(1) > 0 \).

**Consistency of estimator \( \hat{\Psi}_h \)**

**Theorem 0.2.** For some fixed \( \chi \in \mathbb{H} \), assume \( \exists \delta' > \delta > 0 \) such that

(i) \( \frac{2 + \delta}{2 + \delta'} + \frac{1 - \delta/2 + \delta'}{2} \leq 1 \),

(ii) \( E|X_i - \Psi(X_i)|^{2+\delta'} < \infty \),

(iii) \( \sum_j a(j) \frac{\delta'}{\delta} < \infty \),

where \( a(\cdot) \) is the mixing sequence of the coefficient \( \{X_i, i \in \mathbb{N}\} \). Also assume regularity conditions (C1)-(C4) given in [1]. Then

\[
\hat{\Psi}_h(\chi) = \Psi(\chi) - B_h + o_p\left(\frac{1}{\sqrt{n}F_h(h)^{1+\delta'}}\right)
\]

where

\[
B_h = h^\delta \prod_{k=1}^{n-1} V_{X,h}(0)\epsilon_k.
\]

Remark. The assumptions (i)-(iii) show a trade-off between the moment assumptions and the mixing conditions. The conditions on mixing coefficients can be less stringent if higher moments are assumed. The parameter \( \delta' \) controls the moment while \( \delta \) controls the mixing condition.

**REFERENCES**


**SIMULATIONS**

**Figure 1:** 5 Curves \( X_{101}, X_{102}, \ldots, X_{105} \) from the sample.

**Figure 2:** Estimator \( \hat{\Psi}_h(\chi) \) (dashed); true operator \( \Psi(\chi) \) (solid).

**Computing Kernel Estimator.** With the 250 simulated curves, we use a learning sample (the first 200 curves) to compute the kernel estimator \( \hat{\Psi}_h \), and evaluate it against the test sample (the last 50 curves).

**PARAMETER h** is chosen by a standard cross validation procedure, with a fixed semi-metric \( d(X_1, X_2) \) - see reference for details.

**Figure 2** compares the kernel estimation (i.e. \( \hat{\Psi}_h(\chi) \)) with the true operator (i.e. \( \Psi(\chi) \)) at \( \chi = X_{201}, \ldots, X_{204} \).