AN INTRODUCTION TO EXTERIOR FORMS WITH AN APPLICATION TO CAUCHY'S STRESS TENSOR

AN OVERVIEW OF CARTAN'S EXTERIOR FORMS IN MY TEXT "THE GEOMETRY OF PHYSICS"

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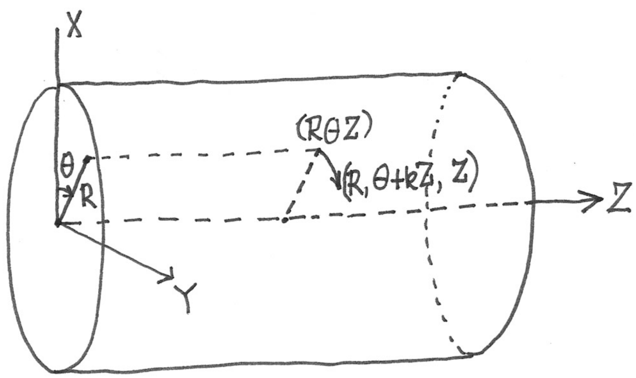
CONTENTS

I am very grateful to my engineering colleague Professor Hidenori Murakami for many very helpful conversations.

0) Introduction. The goal is to introduce **exterior calculus** in a *brief* way that leads directly to their use in engineering and physics, both in basic physical concepts and in specific engineering calculations. The presentation will be very informal. Many times a proof will be omitted so that we can get quickly to a calculation. In some "proofs" we shall look only at a typical term. It will be left to the readers to decide if they are compelled to see more details in my text, *The Geometry of Physics*.

The chief mathematical prerequisites for this overview are sophomore courses in basic linear algebra, partial derivatives, multiple integrals, tangent vectors to parameterized curves, but not necessarily "vector calculus", i.e., curls, divergences, line and surface integrals, Stokes' theorem, ... These last topics will be developed here using Cartan's "exterior calculus".

We shall take advantage of the fact that most mechanical engineers live in Euclidean 3–space $\mathbb{R}^3$ with its everyday metric structure, but we shall try to use methods that make sense in much more general situations. Instead of including exercises we shall consider, in the second part of this exposition, one main example and illustrate *everything* in terms of this example, but hopefully the general principles will be clear. This engineering example, will be the following. Take an elastic circular cylindrical rod of radius a and length L, described in cylindrical coordinates (we shorten "coordinates" to "**coords**") R,Θ, Z, with ends of cylinder at Z=0 and Z=L. Look at this same cylinder except that it has been axially twisted through an angle kZ proportional to the distance Z from the end Z=0.



We shall **neglect gravity** and investigate the **stresses** in the cylinder in its final twisted state, in the first approximation, i.e., where we put $k^2 = 0$. Since "stress" and "strain" are "tensors" (as Cauchy and I will show) this is classically treated via "tensor analysis". The final equilibrium state involves surface integrals and the tensor divergence of the Cauchy stress tensor. Our main tool will **not** be the usual *classical* tensor analysis (Christoffel symbols $\Gamma^i{}_{jk}$ ..,etc.) but rather **exterior differential forms**, first used in the 19^{th} Century (by Grassmann, Poincare′, Volterra, ..., and especially **Elie Cartan**) which, I believe, is a far more appropriate tool.

We are very much at home with cartesian coords but curvilinear coords play a very important role in physical applications, and **the fact that there are two distinct types of vectors** that arise in curvilinear coords (and, even more so, in curved *spaces*) that appear identical in cartesian coords, MUST be understood, not only when making calculations but also in our understanding of the basic ingredients of the physical world. We shall let x^i, and u^i,

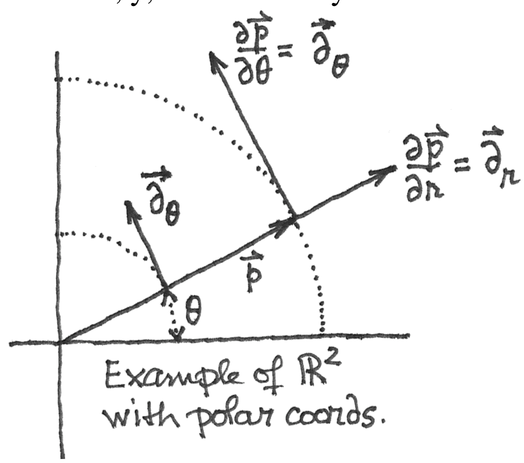
$i=1,2,3$, be **general** (curvilinear) coords, in Euclidean 3 dimensional space $\mathbb{R}^3$. *If cartesian coordinates are wanted I will, say so* ***explicitly*** .

1) Two Kinds of Vectors. There are two kinds of vectors that appear in physical applications and **it is important that we distinguish between them**. First there is the familiar "arrow" version.

Consider n dimensional Euclidean space $\mathbb{R}^n$ with cartesian coordinates $x^1, \dots, x^n$ and local (perhaps curvilinear) coordinates $u^1, \dots, u^n$.

Ex. $\mathbb{R}^2$ with Cartesian coords $x^1=x$, $x^2=y$, and with polar coords $u^1=r$, $u^2=\theta$

Ex. $\mathbb{R}^3$ with Cartesian coords x, y, z and with cylindrical coords R, Θ, Z



Let **p** be the position vector from the origin of $\mathbb{R}^n$ to the point p. In the curvilinear coordinate system u, the coordinate curve C_i through the point p is the curve where all u^j, $j \neq i$, are constants, and where u^i is used as parameter. Then the tangent vector to this curve in $\mathbb{R}^n$ is

$$\partial \mathbf{p}/\partial u^i \qquad \text{which we shall abbreviate to} \qquad \partial_i \quad \text{or} \quad \partial/\partial u^i$$

At the point p these n vectors $\partial_1, \dots, \partial_n$ form a **basis** for all vectors in $\mathbb{R}^n$ based at p. Any vector **v** at p has a unique expansion with curvilinear coord *components* $(v^1, \dots, v^n)$

$$\mathbf{v} = \Sigma_i v^i \partial_i = \Sigma_i \partial_i v^i$$

We prefer the last expression with the components to the *right* of the basis vectors. We can then form the matrices

$$\partial = \text{the row } (\partial_1, \dots, \partial_n) \qquad \text{and} \qquad v = \text{the column } (v^1, \dots, v^n)^T$$

(T denotes transpose) and then we can write the matrix expression (with **v** a 1×1 matrix)

$$\boxed{\mathbf{v} = \partial\, v} \qquad (1)$$

(It is traditional to put the vectorial components in a *column* matrix, as we have.) Please **beware** though that in $\boldsymbol{\partial}_i v^i$ or $(\boldsymbol{\partial}/\partial u^i) v^i$ or $\mathbf{v} = \boldsymbol{\partial}\, v$, **the bold $\boldsymbol{\partial}$ does not differentiate the component term to the right; it is merely the symbol for a basis vector.** Of course we can still differentiate a function f along a vector $\mathbf{v}$ by *defining*

$$\mathbf{v}(f) = (\Sigma_i\ \boldsymbol{\partial}_i v^i)(f) = \Sigma_i \boldsymbol{\partial}/\partial u^i\ (f)\ v^i := \Sigma_i (\partial f/\partial u^i)\, v^i$$

replacing the basis vector $\boldsymbol{\partial}/\partial u^i$ with bold $\boldsymbol{\partial}$ by the partial differential operator $\partial/\partial u^i$ and then applying to the function f. **A vector *is* a first order differential operator on functions**!

In cylindrical coords R,Θ, Z in $\mathbb{R}^3$ we have the basis vectors $\boldsymbol{\partial}_R = \boldsymbol{\partial}/\partial R$, $\boldsymbol{\partial}_\theta = \boldsymbol{\partial}/\partial\Theta$, and $\boldsymbol{\partial}_Z = \boldsymbol{\partial}/\partial Z$.

Let $\mathbf{v}$ be a vector at a point p. **We can always find a curve $u^i = u^i(t)$ through p whose velocity vector there is $\mathbf{v}$**, $v^i = du^i/dt$. Then if u' is a second coordinate system about p, we then have $v'^j = du'^j/dt = (\partial u'^j/\partial u^i)\, du^i/dt = (\partial u'^j/\partial u^i)\, v^i$. Thus the **components** of a vector transform under a change of coordinates by the rule

$$v'^j = \Sigma_i (\partial u'^j/\partial u^i)\, v^i \qquad \text{or as matrices} \qquad v' = (\partial u'/\partial u) v \qquad (2)$$

where $(\partial u'/\partial u)$ is the Jacobian matrix. This is the **transformation law** for the components of a **contravariant** vector, or **tangent** vector, or simply **vector**.

There is a **second, different type of vector**. In linear algebra we learn that to each vector space V (in our case the space of all vectors at a point p) we can associate its **dual** vector space V* of all real **linear** functionals $\alpha: V \to \mathbb{R}$. In coordinates $\alpha(\mathbf{v})$ is a number

$$\alpha(\mathbf{v}) = \Sigma_i a_i v^i$$

We shall explain why i is a *sub*script in a_i shortly.

The most familiar linear functional is the differential of a function df. As a function on vectors it is defined by the derivative of f along v

$$df(\mathbf{v}) := \mathbf{v}(f) = \Sigma_i (\partial f/\partial u^i)\, v^i \qquad \text{and so} \qquad (df)_i = \partial f/\partial u^i$$

Let us write df in a much more familiar form. In elementary calculus there is mumbo–jumbo to the effect that du^i is a function of pairs of points: it gives you the difference in the u^i coordinates between the points, and the points do not need to be close together. What is *really* meant is

$\boxed{du^i \textbf{ is the linear functional that reads off the } i^{th} \textbf{ component of any vector } \mathbf{v}}$
with respect to the basis vectors of the coord system u

$$du^i(\mathbf{v}) = du^i(\Sigma_j \boldsymbol{\partial}_j v^j) := v^i$$

Note that this agrees with $du^i(\mathbf{v}) = \mathbf{v}(u^i)$ since $\mathbf{v}(u^i) = (\Sigma_j \boldsymbol{\partial}_j v^j)(u^i) = \Sigma_j (\partial u^i/\partial u^j)\, v^j$ $= \Sigma_j \delta^i_j v^j = v^i$.

Then we can write

$$df(\mathbf{v}) = \Sigma_i (\partial f/\partial u^i)\, v^i = \Sigma_i (\partial f/\partial u^i)\, du^i(\mathbf{v})$$

i.e.,

$$\boxed{df = \Sigma_i\, (\partial f/\partial u^i)\, du^i}$$

as usual, except that now both sides have meaning as linear functionals on vectors.
Warning. We shall see that this is **not** the gradient vector of f!

It is very easy to see that $du^1, \ldots , du^n$ form a basis for the space of linear functionals at each point of the coordinate system u, since they are linearly independent. In fact, this basis of V* is the **dual basis** to the basis $\partial_1, \ldots, \partial_n$, meaning

$$\boxed{du^i\, (\partial_j) = \delta^i_j}$$

Thus in the coordinate system u, every linear functional α is of the form

$$\boxed{\alpha = \Sigma_i\ a_i\,(u)\, du^i} \quad \text{where} \qquad \alpha(\partial_j) = \Sigma_i\ a_i\,(u)\, du^i\, (\partial_j) = \Sigma_i\ a_i\,(u)\delta^i_j = a_j$$

is the j^{th} component of α.

We shall see in section 8) that it is **not** true that every α is = df for some f!

Corresponding to (1) we can write the matrix expansion for a linear functional as

$$\alpha = (a_1 , \ldots , a_n)\, (du^1 , \ldots , du^n)^T = a\, du \tag{3}$$

i.e., a is a **row** matrix and du is a column matrix!

If V is the space of contravariant vectors at p, then V* is called the space of **covariant** vectors, or covectors, or **1-forms** at p. Under a change of coordinates, using the chain rule, $\alpha = a'du' = a\, du = (a)\, (\partial u/\partial u')(du')$, and so

$$\boxed{a' = a\, (\partial u/\partial u') = a\, (\partial u'/\partial u)^{-1} \qquad \text{i.e.,} \quad a'_j = \Sigma_i\, a_i\ (\partial u^i/\partial u'^j)} \tag{4}$$

which should be compared with (2). This is the law of transformation of components of a covector.

Note that by definition, if α is a covector and $\mathbf{v}$ is a vector, then the value

$$\alpha(\mathbf{v}) = a\, v = \Sigma_i\ a_i\, v^i$$

is **invariant**, i.e., independent of the coordinates used. This also follows, from (2) and (4)

$$\alpha(\mathbf{v}) = a'\, v' = a\, (\partial u/\partial u')\, (\partial u'/\partial u)\, v \ = a\, (\partial u'/\partial u)^{-1}\, (\partial u'/\partial u)\, v = \ a\, v$$

Note that a vector can be considered as a linear functional on covectors, $\mathbf{v}(\alpha) := \alpha(\mathbf{v})$ $= \ \Sigma_i\ a_i\, v^i$.

2) Superscripts, Subscripts, Summation Convention. First the "**summation convention**". Whenever we have a single term of an expression with any number of indices up and down, e.g., $T^{abc}{}_{de}$, if we rename one of the **lower** indices, say d so that it becomes the same as one of the **upper** indices, say b, and if we then sum over this index, the result, call it S,

$$\Sigma_b T^{abc}{}_{be} = S^{ac}{}_e$$

is called a **contraction** of T. The index b has disappeared (it was a summation or "dummy" index on the left expression; you could have called it anything.) This process of summing over a repeated index **that occurs as both a subscript and a superscript** occurs so often that we shall **omit the summation sign** and merely write, e.g., $T^{abc}{}_{be} = S^{ac}{}_e$. This convention does **not** apply to two upper or two lower indices. Here is why.

We have seen that if α is a covector, and if **v** is a vector then $\alpha(\mathbf{v}) = a_i v^i$ is an invariant, independent of coordinates. But if we have another vector, say $\mathbf{w} = \partial\, w$ then $\Sigma_i\, v^i w^i$ **will not be invariant**;

$$\Sigma_i\, v'^i w'^i = v'^T w' = [(\partial u'/\partial u) v]^T (\partial u'/\partial u)\, w = v^T (\partial u'/\partial u)^T (\partial u'/\partial u)\, w$$

will not = $v^T\, w$, for all **v**,**w** unless $(\partial u'/\partial u)^T = (\partial u'/\partial u)^{-1}$, i.e., unless the coordinate change matrix is an **orthogonal** matrix, as it is when u and u' are Cartesian coordinate systems.

Our **conventions** regarding the **components** of vectors and covectors

$$\boxed{\text{(contravariant} \Rightarrow \text{index up) and (covariant} \Rightarrow \text{index down)}} \qquad (**)$$

help us avoid errors! For example, in calculus, the differential equations for curves of **steepest ascent** for a function f are written in cartesian coords as

$$dx^i/dt = \partial f/\partial x^i$$

but these equations cannot be correct , say, in spherical coordinates, since we cannot equate the *contravariant* components v^i of the velocity vector with the *covariant* components of the differential df; **they transform in different ways** under a (non−orthogonal) change of coordinates. We shall see the correct equations for this situation in section 3).

> **Warning**. Our convention (**) applies only to the **components** of vectors and covectors. In $\alpha = a_i dx^i$, the a_i are the components of a single covector α, while each individual dx^i is itself a basis covector, **not** a component. The summation convention, however, always holds.

I cringe when I see expressions like $\Sigma_i\, v^i w^i$ in non−cartesian coords, for the notation is informing me that I have misunderstood the "variance" of one of the vectors.

3) Riemannian Metrics. One *can* identify vectors and covectors by introducing an *additional* structure, but the identification will depend on the structure chosen. The metric structure of ordinary Euclidean space $\mathbb{R}^3$ is based on the fact that we can measure angles and lengths of vectors and scalar products < , >. The arc length of a curve C is

$$\int_C ds$$

where $ds^2 = dx^2 + dy^2 + dz^2$ in **cartesian** coords. In curvilinear coords u we have, putting $dx^k = (\partial x^k/\partial u^i)\, du^i$, and then

$$\boxed{ds^2 = \Sigma_k (dx^k)^2 = \Sigma_{i,j}\, g_{ij}\, du^i du^j = g_{ij}\, du^i du^j} \qquad (5)$$

where

$$g_{ij} = \Sigma_k\ (\partial x^k/\partial u^i)\ (\partial x^k/\partial u^j) = \text{(since the x coords are cartesian)}\ \ \langle \partial \mathbf{p}/\partial u^i, \partial \mathbf{p}/\partial u^j \rangle$$

$$\boxed{\begin{array}{l} g_{ij} = \langle \partial_i , \partial_j \rangle = g_{ji} \\ \text{and generally} \qquad\qquad (6) \\ \langle \mathbf{v},\mathbf{w} \rangle = g_{ij}\, v^i\, w^j \end{array}}$$

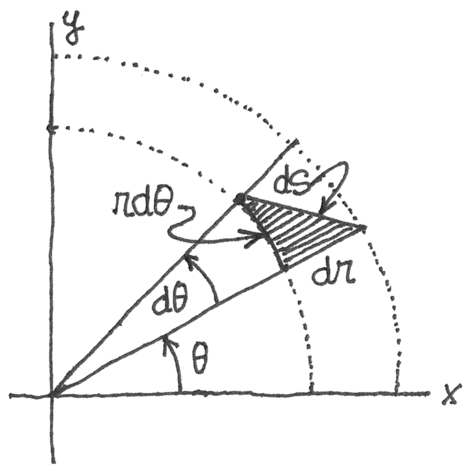
For example, consider the plane $\mathbb{R}^2$ with cartesian coords $x^1 = x$, $x^2 = y$, and polar coords $u^1 = r$, $u^2 = \theta$. Then

$$\begin{bmatrix} g_{xx} = 1 & g_{xy} = 0 \\ g_{yx} = 0 & g_{yy} = 1 \end{bmatrix} \text{ i.e., } \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, from $x = r\cos\theta$, $dx = dr\cos\theta - r\sin\theta\, d\theta$, etc., we get $ds^2 = dr^2 + r^2\, d\theta^2$,

$$\begin{bmatrix} g_{rr} = 1 & g_{r\theta} = 0 \\ g_{\theta r} = 0 & g_{\theta\theta} = r^2 \end{bmatrix} \text{ i.e., } \begin{bmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\vartheta\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \qquad (7)$$

which is "evident" from the picture



In **spherical** coords a picture shows $ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2$, where ϑ is co–latitude and φ is – longitude, so $(g_{ij}) = \mathrm{diag}(1,\ r^2,\ r^2\sin^2\vartheta)$. In **cylindrical** coords, $ds^2 = dR^2 + R^2 d\Theta^2 + dZ^2$, with $(g_{ij}) = \mathrm{diag}(1,\ R^2,\ 1)$.

Let us look again at the expression (5). If α and β are 1–forms, i.e., linear functionals, **define** their **tensor product** $\alpha\otimes\beta$ to be the function of (ordered) **pairs** of vectors defined by

$$\boxed{\alpha\otimes\beta\ (\mathbf{v},\ \mathbf{w}) := \alpha(\mathbf{v})\beta(\mathbf{w})} \qquad (8)$$

In particular

$$(du^i \otimes du^k)(\mathbf{v},\ \mathbf{w}) := v^i\, w^k$$

Likewise $(\partial_i \otimes \partial_j)\,(\alpha, \beta) = a_i\, b_j$ (why?)

$\alpha\otimes\beta$ is a *bilinear* function of $\mathbf{v}$ and $\mathbf{w}$, i.e., it is linear in each when the other is unchanged. A **second rank covariant tensor** is just such a bilinear function and in the coord system u it can be expressed as

$$\Sigma_{i,j}\, a_{ij}\, du^i \otimes du^j$$

where the coefficient matrix (a_{ij}) is written with indices down. **Usually the tensor product sign ⊗ is omitted** (in $du^i \otimes du^j$ but *not* in $\alpha\otimes\beta$). For example, the metric

$$\boxed{ds^2 = g_{ij}\, du^i \otimes du^j = g_{ij}\, du^i\, du^j} \qquad (5')$$

is a second rank covariant tensor that is **symmetric**, i.e., $g_{ji} = g_{ij}$. We may write

$$ds^2(\mathbf{v},\mathbf{w}) = \langle \mathbf{v},\mathbf{w}\rangle$$

It is easy to see that under a change of coords $u' = u'(u)$, demanding that ds^2 be independent of coords yields the transformation rule

$$g'_{ab} = (\partial u^i/\partial u'^a)\, g_{ij}\, (\partial u^j/\partial u'^b) \qquad (9)$$

for the components of a 2^{nd} rank *covariant* tensor.

Remark. We have been using the euclidean metric structure to construct (g_{ij}) in any coord system, but there are times when other structures are more appropriate. For example, when considering some delicate astronomical questions, a metric from Einstein's general relativity yields more accurate results. When dealing with complex analytic functions in the upper half plane $y > 0$, Poincare′ found that the planar metric $ds^2 = (dx^2+dy^2)/y^2$ was very useful. In general, when some 2^{nd} rank covariant tensor (g_{ij}) is used in a metric $ds^2 = g_{ij}\, dx^i dx^j$, (in which case it must be symmetric and positive definite), this metric is called a **Riemannian metric**, after B. Riemann, who was the first to consider this generalization of **Gauss**' thoughts.

Given a Riemannian metric, one can associate to each (contravariant) vector $\mathbf{v}$ a covector ν by

$$\nu(\mathbf{w}) = \langle \mathbf{v},\mathbf{w}\rangle$$

for *all* vectors $\mathbf{w}$, i.e.,

$$\nu_j\, w^j = v^k g_{kj}\, w^j \qquad \text{and so} \qquad \nu_j = v^k g_{kj} = g_{jk}\, v^k$$

In *components*, it is traditional to use the same letter for the covector as for the vector

$$\boxed{v_j = g_{jk}\, v^k}$$

there being no confusion since the covector has the subscript. We say that *we lower the contravariant index* by means of the covariant metric tensor (g_{jk}).

Similarly, since (g_{jk}) is the matrix of a positive definite quadratic form ds^2, it has an inverse matrix, written (g^{jk}), which can be shown to be a **contravariant** 2^{nd} rank symmetric tensor (a bilinear function of pairs of covectors given by $g^{jk}\, a_j\, b_k$). Then for each covector α we can associate a vector $\mathbf{a}$ by $a^i = g^{ij}\, a_j$, i.e., we *raise the covariant index* by means of the contravariant metric tensor (g^{jk}).

The **gradient vector** of a function f is defined to be the vector **grad** $f = \nabla f$ associated to the covector df, i.e., $df(\mathbf{w}) = \langle \nabla \mathbf{f},\mathbf{w}\rangle$

$$(\nabla f)^i := g^{ij}\, \partial f/\partial u^j$$

Then the correct version of the equations of steepest ascent considered at the end of section 2) is

$$du^i/dt = (\nabla f)^i = g^{ij}\,\partial f/\partial u^j$$

in *any* coords. For example, in polar coords, from (7), we see $g^{rr}=1$, $g^{\theta\theta}=1/r^2$, $g^{r\theta}=0=g^{\theta r}$.

4) **Tensors**. We shall consider examples rather than generalities.

(i) A tensor of the 3rd rank, twice contravariant, once covariant, is locally of the form

$$A = \partial_i \otimes \partial_j\, A^{ij}{}_k \otimes du^k$$

It is a trilinear function of pairs of covectors $\alpha = a_i\, du^i$, $\beta = b_j du^j$, and a single vector $\mathbf{v} = \partial_k\, v^k$

$$A(\alpha, \beta, \mathbf{v}) = a_i\, b_j\, A^{ij}{}_k\, v^k \qquad \text{summed, of course, on all indices.}$$

It's components transform as $\qquad A'^{\,ef}{}_g = (\partial u'^e/\partial u^i)\,(\partial u'^f/\partial u^j)\,A^{ij}{}_k\,(\partial u^k/\partial u'^g)$

If we **contract** on i and k, the result $B^j := A^{ij}{}_i$ are the components of a contravariant **vector**
$B'^f = A'^{ef}{}_e = A^{ij}{}_k(\partial u'^f/\partial u^j)(\partial u^k/\partial u'^e)(\partial u'^e/\partial u^i) = A^{ij}{}_k\,(\partial u'^f/\partial u^j)\,\delta^k{}_i = A^{ij}{}_i\,(\partial u'^f/\partial u^j) = (\partial u'^f/\partial u^j)\,B^j$.

(ii) A **linear transformation** is a 2nd rank ("mixed") tensor $P = \partial_i P^i{}_j \otimes du^j$. Rather than thinking of this as a real valued bilinear function of a covector and a vector, we usually consider it as a **linear function taking vectors into vectors**, (called a vector valued 1–form in 13)) $\qquad P(\mathbf{v}) = [\partial_i P^i{}_j \otimes du^j]\,(\mathbf{v}) := \partial_i\, P^i{}_j\,\{du^j(\mathbf{v})\} = \partial_i\, P^i{}_j\, v^j$

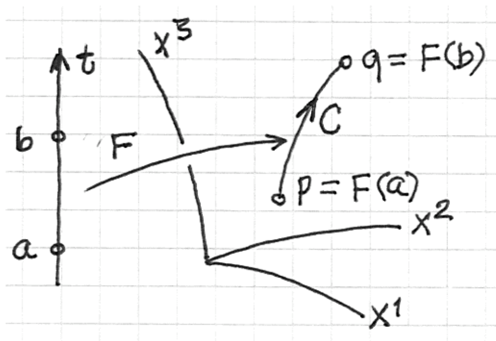
i.e., the usual $\qquad [P(\mathbf{v})]^i = P^i{}_j\, v^j$

Under a change of coords the matrix $(P^i{}_j)$ transforms as $P' = (\partial u'/\partial u)\,P\,(\partial u'/\partial u)^{-1}$, as usual. If we contract we obtain a scalar (invariant), $\mathrm{tr}\,P := P^i{}_i$, the trace of P. $\mathrm{tr}\,P' = \mathrm{tr}\,P\,(\partial u'/\partial u)^{-1}(\partial u'/\partial u) = \mathrm{tr}\,P$

Beware. If we have a twice covariant tensor G (a "bilinear form"), e.g., a metric (g_{ij}), then $\Sigma_k\, g_{kk}$ is **not a scalar**, although it is the trace of the matrix; see e.g., equation (7). This is because the transformation law for the matrix G is, from (9), $G' = (\partial u/\partial u')^T\, G\,(\partial u/\partial u')$ and $\mathrm{tr}\,G' \neq \mathrm{tr}\,G$ generically.

INTEGRALS and EXTERIOR FORMS

5) **Line Integrals**. We illustrate in $\mathbb{R}^3$ with any coords x. For simplicity, let C be a smooth "oriented" or "directed" curve, the image under $F : [a,b] \subset \mathbb{R}^1 \to C \subset \mathbb{R}^3$, (which is read " F maps the interval [a,b] on $\mathbb{R}^1$ into the curve C in $\mathbb{R}^3$") with F(a)= some p and F(b) = some q.



If $\alpha = \alpha^1 = a_i(x)\, dx^i$ is a 1–form, a covector, in $\mathbb{R}^3$, we **define** the line integral $\int_C \alpha$ as follows. Using the parameterization $x^i = F^i(t)$ of C, we define

$$\boxed{\int_C \alpha^1 = \int_C a_i(x)\, dx^i := \int_a^b a_i(x(t))\, (dx^i/dt)\, dt = \int_a^b \alpha(d\mathbf{x}/dt)\, dt} \qquad (10)$$

We say that we *pull back* the form α^1 (that lives in $\mathbb{R}^3$) to a 1–form on the parameter space $\mathbb{R}^1$, called the **pull–back** of α, denoted by $F^*(\alpha)$

$$F^*(\alpha) = \alpha(d\mathbf{x}/dt)\, dt = a_i(x(t))\, (dx^i/dt)\, dt$$

and then take the *ordinary* integral $\int_a^b \alpha(d\mathbf{x}/dt)\, dt$. It is a classical theorem that the result is **independent of the parameterization** of C chosen, so long as the resulting curve has the same direction. This will become "apparent" from the usual geometric interpretation that we now present.

In the definition **there has been no mention of arc length or scalar product**. Suppose now that a Riemannian metric (e.g., the usual metric in $\mathbb{R}^3$) is available. Then to α we may associate the vector $\mathbf{A}$. Then $\alpha(d\mathbf{x}/dt) = \langle \mathbf{A}, d\mathbf{x}/dt\rangle = \|\mathbf{A}\|\, \|d\mathbf{x}/dt\| \cos\angle\, (\mathbf{A}, d\mathbf{x}/dt)$. But, from (5')

$$\|d\mathbf{x}/dt\| := [g_{ij}\, (dx^i/dt)(dx^j/dt)]^{1/2} = (ds/dt) \qquad \text{and thus} \qquad F^*(\alpha) = A_t\, (ds/dt)dt$$

$$\boxed{\int_C \alpha = \int_C A_t\, ds} \qquad (11)$$

the usual integral with respect to arc length of the tangential component of $\mathbf{A}$ along C. This "shows" independence of parameter t chosen, **but to evaluate the integral one would usually just use** (10) **which involves no metric at all!**

Moral. The integrand in a line integral is naturally a **1–form**, *not* a vector.

For example, in **any** coords, **force is often a 1–form** f^1 since a basic measure of force is given by a line integral $W = \int_C f^1 = \int_C f_k\, dx^k$ which measures the **work** done by the force along the curve C, and **this does not require a metric**. Frequently there is a force **potential** V such that $f^1 = dV$, exhibiting f *explicitly* as a covector. (In this case, from (10), $W = \int_C f^1 = \int_C dV = \int_a^b dV(d\mathbf{x}/dt)\, dt = \int_a^b (\partial V/\partial x^i)\, (dx^i/dt)\, dt = \int_a^b \{dV(\mathbf{x}(t)/dt\}\, dt = V[\mathbf{x}(b)] - V[\mathbf{x}(a)] = V(q) - V(p)$.)
Of course metrics do play a large role in mechanics. In Hamiltonian mechanics, a particle of mass m has a kinetic energy $T = mv^2/2 = m\, g_{ij} \dot{x}^i \dot{x}^j/2$ (where $\dot{x}^i$ is dx^i/dt) and its **momentum** is defined, when the potential energy is independent of $d\mathbf{x}/dt$, by $p_k := \partial T/\partial \dot{x}^k =$
$(1/2)m\, g_{ij}\, (\delta^i{}_k \dot{x}^j + \dot{x}^i \delta^j{}_k) = (m/2)\, (g_{kj}\, \dot{x}^j + g_{ik}\, \dot{x}^i) = m\, g_{kj}\, \dot{x}^j$. Thus p is m times the **covariant version of the velocity vector** $d\mathbf{x}/dt$.

The momentum 1–form "$p_i\, dx^i$" on the "phase space" with coords (x, p) plays a *central role* in all of Hamiltonian mechanics.

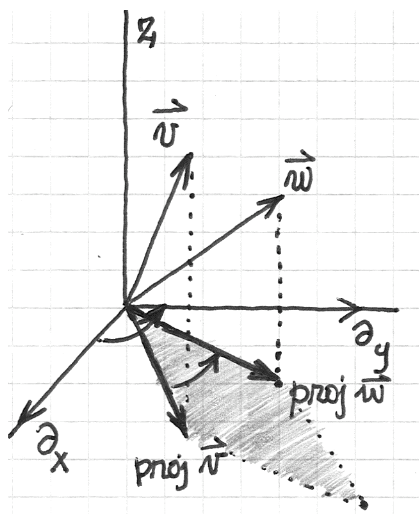
6) **Exterior 2–Forms**.We have already defined the **tensor** product $\alpha^1 \otimes \beta^1$ of two 1–forms to be the bilinear form $\alpha^1 \otimes \beta^1(\mathbf{v},\mathbf{w}) = \alpha^1(\mathbf{v})\beta^1(\mathbf{w})$. We now define a more **geometrically** significant **wedge** or **exterior product** $\alpha \wedge \beta$ to be the *skew symmetric* bilinear form

$$\alpha^1 \wedge \beta^1 := \alpha^1 \otimes \beta^1 - \beta^1 \otimes \alpha^1$$

$$du^j \wedge du^k(\mathbf{v},\mathbf{w}) = v^j w^k - v^k w^j = \begin{vmatrix} du^j(\mathbf{v}) & du^j(\mathbf{w}) \\ du^k(\mathbf{v}) & du^k(\mathbf{w}) \end{vmatrix} \tag{12}$$

In **cartesian** coords x,y,z in $\mathbb{R}^3$ (see figure below)

$dx \wedge dy\, (\mathbf{v},\mathbf{w}) = \pm$ area of parallelogram spanned by the projections of $\mathbf{v}$ and $\mathbf{w}$ into the x–y plane, the plus sign used only if proj($\mathbf{v}$) and proj($\mathbf{w}$) describe the same orientation of the plane as the basis vectors ∂_x and ∂_y.



Let now x^i, i=1,2,3 be **any** coords in $\mathbb{R}^3$. Note that

$$\boxed{dx^j \wedge dx^k = -\, dx^k \wedge dx^j \quad \text{and} \quad dx^k \wedge dx^k = 0 \quad \text{(no sum!)}} \tag{13}$$

The most general **exterior 2–form** is of the form $\beta^2 = \sum_{i<j} b_{ij}\, dx^i \wedge dx^j$ where $b_{ji} = -\, b_{ij}$. In $\mathbb{R}^3$ $\beta^2 = b_{12} dx^1 \wedge dx^2 + b_{23}\, dx^2 \wedge dx^3 + b_{13}\, dx^1 \wedge dx^3$, or, as we prefer, for reasons soon to be evident

$$\boxed{\beta^2 = b_{23}\, dx^2 \wedge dx^3 + b_{31}\, dx^3 \wedge dx^1 + b_{12} dx^1 \wedge dx^2} \tag{14}$$

An exterior 2–form is a skew symmetric covariant tensor of the 2nd rank in the sense of section 4). We frequently will omit the *term* "exterior", but **never** the wedge $\wedge$.

(7) **Exterior p–Forms and Algebra in $\mathbb{R}^n$.** The **exterior algebra** has the following properties.We have already discussed 1–forms and 2–forms. An (*exterior*) p–form α^p in $\mathbb{R}^n$ is a completely skew symmetric multilinear function of p–tuples of vectors $\alpha(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ that changes sign whenever two vectors are interchanged. In any coords x, e.g., the 3–form $dx^i \wedge dx^j \wedge dx^k$ in $\mathbb{R}^n$ is defined by

$$dx^i \wedge dx^j \wedge dx^k(\mathbf{A}, \mathbf{B}, \mathbf{C}) := \begin{vmatrix} dx^i(\mathbf{A}) & dx^i(\mathbf{B}) & dx^i(\mathbf{C}) \\ dx^j(\mathbf{A}) & dx^j(\mathbf{B}) & dx^j(\mathbf{C}) \\ dx^k(\mathbf{A}) & dx^k(\mathbf{B}) & dx^k(\mathbf{C}) \end{vmatrix} = \begin{vmatrix} A^i & B^i & C^i \\ A^j & B^j & C^j \\ A^k & B^k & C^k \end{vmatrix} \tag{15}$$

When the coords are Cartesian the interpretation of this is similar to that in (12). Take the three vectors at a given point x in $\mathbb{R}^n$, project them down into the 3 dimensional affine subspace of $\mathbb{R}^n$ spanned by ∂_i, ∂_j, and ∂_k at x, and read off ± the 3–volume of the parallelopiped spanned by the projections, the + used only if the projections define the same orientation as ∂_i, ∂_j, and ∂_k.

Clearly any interchange of a single pair of dx's will yield the negative, and thus **if the same dx^i appears twice the form will vanish**, just as in (12). Similarly for a p–form. The most general 3–form is of the form $\alpha^3 = \sum_{i<j<k} a_{ijk}\, dx^i \wedge dx^j \wedge dx^k$. In $\mathbb{R}^3$ there is only one non–vanishing 3–form, $dx^1 \wedge dx^2 \wedge dx^3$ and its multiples. In **cartesian** coords this is the **volume form** vol^3, but in spherical coords we know that $dr \wedge d\theta \wedge d\phi$ does **not** yield the Euclidean volume element, which is $r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi$. We will discuss this soon. Note further that all p>n forms in $\mathbb{R}^n$ vanish since there are always repeated dx's in each term.

We take the **exterior product of a p–form α and a q–form β**, yielding a p+q form $\alpha \wedge \beta$ by expressing them in terms of the dx's, using the usual algebra (**including the associative** law), except that the product of dx's is anti–commutative, $dx \wedge dy = - dy \wedge dx$. For examples in $\mathbb{R}^3$ with any coords

$$\alpha^1 \wedge \gamma^1 = (a_1 dx^1 + a_2 dx^2 + a_3 dx^3) \wedge (c_1 dx^1 + c_2 dx^2 + c_3 dx^3) = \ldots (a_2 dx^2) \wedge (c_1 dx^1) + \ldots + (a_1 dx^1) \wedge (c_2 dx^2) + \ldots$$
$$= (a_2 c_3 - a_3 c_2)\, dx^2 \wedge dx^3 + (a_3 c_1 - a_1 c_3)\, dx^3 \wedge dx^1 + (a_1 c_2 - a_2 c_1)\, dx^1 \wedge dx^2$$

which in **cartesian** coords has the components of the vector product $\mathbf{a} \times \mathbf{c}$. Also we have

$$\alpha^1 \wedge \beta^2 = (a_1 dx^1 + a_2 dx^2 + a_3 dx^3) \wedge (b_{23}\, dx^2 \wedge dx^3 + b_{31}\, dx^3 \wedge dx^1 + b_{12}\, dx^1 \wedge dx^2)$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3)\, dx^1 \wedge dx^2 \wedge dx^3$$

(where we use the notation $b_1 := b_{23}$, $b_2 = b_{31}$, $b_3 = b_{12}$, but **only** in cartesian coords) with component $\mathbf{a} \bullet \mathbf{b}$ in cartesian coords. **The $\wedge$ product in Cartesian $\mathbb{R}^3$ yields both the dot • and the cross × products of vector analysis!! The • and × products of vector analysis have strange expressions when curvilinear coords are used in $\mathbb{R}^3$, but the form expressions $\alpha^1 \wedge \beta^2$ and $\alpha^1 \wedge \gamma^1$ are always the same**. Furthermore, the × product is nasty since it is not associative, $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$.

By counting the number of interchanges of pairs of dx's one can see the **commutation rule**

$$\alpha^p \wedge \beta^q = (-1)^{pq}\, \beta^q \wedge \alpha^p \tag{16}$$

8) The Exterior Differential d. First a remark. If $\mathbf{v} = \boldsymbol{\partial}_a v^a$ is a contravariant vectorfield, then generically $(\partial v^a/\partial x^b) = Q^a{}_b$ **do not yield the components of a tensor** in curvilinear coords, as is easily seen from looking at the transformation of Q under a change of coords and using (2). It is, however, always possible, in $\mathbb{R}^n$ and in any coords, to take a very important "**exterior**"derivative d **of p−forms**. We define $d\alpha^p$ to be a p+1 form, as follows; α is a sum of forms of the type $a(x)\, dx^i \wedge dx^j \wedge \ldots \wedge dx^k$. Define

$$d\,[dx^i \wedge dx^j \wedge \ldots \wedge dx^k] = 0 \qquad \text{and}$$

$$d\,[a(x)\, dx^i \wedge dx^j \wedge \ldots \wedge dx^k] = da \wedge dx^i \wedge dx^j \wedge \ldots \wedge dx^k \tag{17}$$

$$= \Sigma_r(\partial a/\partial x^r)\, dx^r \wedge dx^i \wedge dx^j \wedge \ldots \wedge dx^k$$

and sum over all the terms in α^p. In particular, in $\mathbb{R}^3$ in **any** coords,

$$df^0 = df = (\partial f/\partial x^1)dx^1 + (\partial f/\partial x^2)dx^2 + (\partial f/\partial x^3)dx^3$$

$$d\alpha^1 = d\,(a_1 dx^1 + a_2 dx^2 + a_3 dx^3) = (\partial a_1/\partial x^2)dx^2 \wedge dx^1 + (\partial a_1/\partial x^3)dx^3 \wedge dx^1 + \ldots =$$

$$= [(\partial a_3/\partial x^2) - (\partial a_2/\partial x^3)]dx^2 \wedge dx^3 + [(\partial a_1/\partial x^3) - (\partial a_3/\partial x^1)]dx^3 \wedge dx^1$$

$$+ [(\partial a_2/\partial x^1) - (\partial a_1/\partial x^2)]dx^1 \wedge dx^2 \tag{18}$$

$$d\beta^2 = d\,(b_1 dx^2 \wedge dx^3 + b_2\, dx^3 \wedge dx^1 + b_3\, dx^1 \wedge dx^2)$$

$$= [(\partial b_1/\partial x^1) + (\partial b_2/\partial x^2) + (\partial b_3/\partial x^3)]\, dx^1 \wedge dx^2 \wedge dx^3$$

In **cartesian** coords we then have correspondences with vector analysis

$$df^0 \Leftrightarrow \nabla \mathbf{f} \bullet \mathbf{dx} \qquad d\alpha^1 \Leftrightarrow (\text{curl}\ \mathbf{a}) \bullet \text{"}\mathbf{dA}\text{"} \qquad d\beta^2 \Leftrightarrow \text{div}\ \mathbf{B}\ \text{"dvol"} \qquad (19$$

the quotes, e.g., "**dA**" being used since this is not really the differential of a 1−form. We shall make this correspondence precise, in any coordinates, later. Exterior differentiation of **exterior forms** does essentially grad, curl and divergence with a **single general formula** (17)!! Also, this machinery works in $\mathbb{R}^n$ as well. Furthermore, **d does not require a metric**. On the other hand, Without a metric (and hence without cartesian coords), **one cannot take the curl of a contravariant vector**. Also to take the **divergence of a vector** requires at least a specified "volume form". These will be discussed in more detail later.

There are two fairly easy but very important properties of the differential d.

$$d^2\alpha^p := d\, d\, \alpha^p = 0 \qquad \text{corresponding to} \qquad \text{curl grad}=0 \quad \text{and div curl} = 0$$

$$\text{and} \tag{20}$$

$$d(\alpha^p \wedge \beta^q) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

For example, in $\mathbb{R}^3$ with function (0–form) f, df= $(\partial f/\partial x)dx + (\partial f/\partial y)dy + (\partial f/\partial z)dz$, and then $d^2 f = (\partial^2 f/\partial x\, \partial y)\, dy\wedge dx + \ldots + (\partial^2 f/\partial y\, \partial x)\, dx\wedge dy + \ldots = 0$, since $(\partial^2 f/\partial y\, \partial x) = (\partial^2 f/\partial x\, \partial y)$. Note then that **a *necessary* condition for a p–form β^p to be the differential of a (p−1)–form α, $\beta^p = d\alpha^{p-1}$, is that $d\beta = d\, d\alpha = 0$**. (What does this say in vector analysis in $\mathbb{R}^3$?) Also, we know that in $\mathbb{R}^3$ $\alpha^1\wedge\beta^1$ is a 2–form $\Leftrightarrow$ $\mathbf{a}\times\mathbf{b}$, $d(\alpha\wedge\beta) \Leftrightarrow$ (from (19) div $\mathbf{a}\times\mathbf{b}$, $d\alpha \Leftrightarrow$ curl $\mathbf{a}$, and we know $\alpha^1 \wedge \gamma^2 = \gamma^2\wedge\alpha^1 \Leftrightarrow \mathbf{a}\bullet\mathbf{c}$. Then (20), in Cartesian coords, says **immediately** that $d(\alpha\wedge\beta) = d\alpha\wedge\beta - \alpha\wedge d\beta$, i.e.,

$$\text{div}\, \mathbf{a}\times\mathbf{b} = (\text{curl}\, \mathbf{a})\bullet\mathbf{b} - \mathbf{a}\bullet(\text{curl}\, \mathbf{b}) \tag{21}$$

9) **Pull–backs**. Let F: $\mathbb{R}^k \to \mathbb{R}^n$ be any differentiable map of k–space into n–space, where any values of k and n are permissible. Let $(u^1,.., u^k)$ be any coords in $\mathbb{R}^k$, let $(x^1, ..,x^n)$ be any coordinates in $\mathbb{R}^n$. Then F is described by n functions $x^i = x^i\,(u^1,..,u^r,.. u^k)$. The pull–back of a function (0–form) $\phi = \phi(x)$ on $\mathbb{R}^n$ is the function $F^*\phi = \phi(x(u)$ on $\mathbb{R}^n$, i.e., the function on $\mathbb{R}^k$ whose value at u is simply the value of ϕ at x=F(u).

Let β^p be a p–form in $\mathbb{R}^n$. We define the **pull–back** $F^*\beta$ to be the p–form in $\mathbb{R}^k$ defined just by using the chain rule and (13). We illustrate with a typical term for a 2=form β^2 in $\mathbb{R}^n$ pulled back to $\mathbb{R}^3$. We demand, thinking of b_{12} as a function on $\mathbb{R}^n$

$$F^*\,[\, b_{12}(x)\, dx^1\wedge dx^2]: =[\, F^*b_{12}(x)]\, [\, F^*\, dx^1] \wedge [F^*dx^2]$$
$$:= b_{12}(x(u))\, [\, (\partial x^1/\partial u^a)\, du^a] \wedge [\, (\partial x^2/\partial u^c)\, du^c]$$

Now $(\partial x^1/\partial u^a)\, du^a = (\partial x^1/\partial u^1)\, du^1 + (\partial x^1/\partial u^2)\, du^2 + (\partial x^1/\partial u^3)\, du^3$ with a similar expression for $(\partial x^2/\partial u^c)\, du^c$. Taking their $\wedge$ product and using (13) (do it!) will yield

$$F^*[\, b_{12}(x)\, dx^1\wedge dx^2] = b_{12}(x(u)) \;\; \Sigma_{a<c}\, [\, \partial(x^1,x^2)/\, \partial(u^a,u^c)]\, du^a\wedge du^c \tag{22}$$

where $\partial(x,y)/\partial(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$ is the usual **Jacobian determinant**.

This procedure plays a key role in discussing surface integrals, see (25). Note that by our construction $F^*\, dx^i = (\partial x^i/\partial u^a)\, du^a = d\, F^*\,(x^i)$, and it can be shown more generally that

$$F^*\,(d\alpha^p) = d\,(F^*\,\alpha^p) \quad \text{and} \quad F^*(\,\alpha\wedge\beta) = F^*(\alpha)\wedge F^*(\beta) \tag{23}$$

(20) and (23) are what make forms so powerful and useful, compared to vector fields. In general, and for our example of $F:\mathbb{R}^2 \to \mathbb{R}^3$, we can define the **push forward** F_* of a vector at a single point u, e.g.,

$$F_* \, \partial/\partial u^a := \Sigma_i \, (\partial x^i/\partial u^a) \, \partial/\partial x^i \, ,$$

yielding a vector at F(u), but we can't map the vector*field* $\partial/\partial u^a$ into a vectorfield in $\mathbb{R}^3$ if, e.g., there are distinct points u and u' such that F(u) =F(u'), because generically the two resulting push forwards, one from u and one from u', won't agree at the image point. Also if the image space dimension n is greater than that of the source space k, then the push forward vectors will not be defined on all of $\mathbb{R}^n$. However, **the pull–*back* of a p–form field is always a p–form field**.

The geometric meaning of the push forward $F_* \mathbf{v}$ is the following. At a point u the vector $\mathbf{v}(u)$ is the velocity vector to some curve C through u, $\mathbf{v}= du/dt$. Then $F_* \mathbf{v}$ **is the velocity vector to the image curve** F(C), $dx(u(t))/dt = (\partial x/\partial u)\, du/dt$, i.e., $dx^i/dt = (\partial x^i/\partial u^a)\, du^a/dt$.

It is a fact that

$$\boxed{F^*\alpha^p (\mathbf{v}, \ldots \mathbf{w}) = \alpha^p(F_*\mathbf{v},\ldots, F_*\mathbf{w})} \qquad (24)$$

which supplies the geometric meaning of the pull–back F*, namely, **the pull–back of α, applied to a p–tuple of vectors at u, has the same value as the original form α on the push forward of these vectors to F(u)**. In fact, using (24) we can define the pullback of any covariant tensor of rank p, not just p–forms. In particular, as we shall see, the pull back of the metric tensor is the principal ingredient of the "strain tensor" in elasticity.

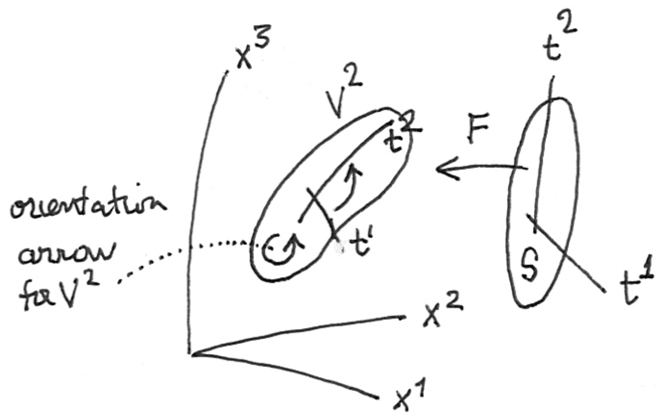
Note that when $F: \mathbb{R}^n \to \mathbb{R}^n$ is the **identity** map, using two sets of coordinates, e.g., $x= r \cos\theta$, $y= r \sin\theta$, then the pull–back $F^*\alpha$ **is simply expressing the form** α ,given in coords x, **in terms of the new coords** u.

10) **Surface Integrals and "Stokes' theorem".** We illustrate with a surface V^2 in $\mathbb{R}^3$. Assume, e.g., that $\mathbb{R}^3$ has the "right handed orientation". Assume that V^2 is also "oriented" meaning that at each point p of V there is a preferred sense of rotation of the tangent plane at p, (indicated in the figure below by a circular arrow), and this sense varies continuously on V. For example, if V has a continuous choice of normal vector everywhere (unlike a Möbius band) then the right hand rule for $\mathbb{R}^3$ will yield an orientation for V.

We are going to define $\int_V \beta^2$ for any 2–form β. If V is sufficiently small we may choose a parameterization of V that yields the same orientation as V, i.e., we ask for a smooth 1–1 map

$$F : \text{region } S^2 \subset \text{some } \mathbb{R}^2 \to \text{onto } V^2 \subset \mathbb{R}^3 \qquad x^i = x^i(t^1,t^2)$$

(If V is too large for such a parameterization, break it up into smaller pieces and add up the individual resulting integrals.) We picture the resulting t^1, t^2 coordinate curves on V as engraved on V just as latitude and longitude curves are engraved on globes of the Earth. We demand that the sense of rotation from the engraved t^1 curve to the t^2 curve on V (i.e, from $F_* \partial_1$ to $F_*\partial_2$) is the same as the given orientation arrow on V. We say V=F(S).



We now define

$$\int_V b_{23}\, dx^2 \wedge dx^3 + b_{31}\, dx^3 \wedge dx^1 + b_{12} dx^1 \wedge dx^2 = \boxed{\int_V \beta^2 = \int_{F(S)} \beta^2 := \int_S F^*\beta}$$

reducing the problem to defining the integral of the pull–back of β over S. First write this out, but for simplicity we just look at the term $b_{31}(x)\, dx^3 \wedge dx^1$. From (22)

$$\int_S F^*(b_{31}(x)\, dx^3 \wedge dx^1) := \int_S b_{31}(x(t))[(\partial x^3/\partial t^a)dt^a \wedge (\partial x^1/\partial t^b)dt^b]$$

$$= \int_S b_{31}(x(t))[\partial(x^3,x^1)/ \partial(t^1,t^2)]dt^1 \wedge dt^2 := \int_S b_{31}(x(t))[\partial(x^3,x^1)/ \partial(t^1,t^2)]dt^1dt^2$$

and where the very last integral, with no ∧, is the **usual double integral** over a region S in the t^1t^2 plane. Thus

$$\boxed{\begin{aligned} &\int_V \beta^2 = \int_{F(S)} \beta^2 = \int_S F^*\beta^2 \qquad (25)\\ &:= \int_S\{b_{23}(x(t))[\partial(x^2,x^3)/\partial(t^1,t^2)] + b_{31}(x(t))[\partial(x^3,x^1)/\partial(t^1,t^2)] + b_{12}(x(t))\, [\partial(x^1,x^2)/\partial(t^1,t^2)]\}\, dt^1dt^2 \end{aligned}}$$

Note that one needn't remember this. **One merely uses the chain rule in calculus and $dt^1 \wedge dt^2 = -dt^2 \wedge dt^1$ to get an integral over a region in the t^1,t^2 plane, then omit the ∧ and evaluate the resulting double integral**.

Interpretation. In **cartesian** coords with the usual metric in $\mathbb{R}^3$, associate to β^2 the vector

$$\mathbf{B} = (B^1=b_{23}, B^2=b_{31}, B^3=b_{12})^T.$$

$\mathbf{n} = [\partial\mathbf{x}/\partial t^1] \times [\partial\mathbf{x}/\partial t^2]$ is a normal to the surface with components

$$([\partial(x^2,x^3)/\partial(t^1,t^2)],\ [\partial(x^3,x^1)/\partial(t^1,t^2)],\ [\partial(x^1,x^2)/\partial(t^1,t^2)])^T.$$

Just as in the case of a curve, where $\| d\mathbf{x}/dt\|\, dt$ is the element of arc length ds, so in the case of a surface, where $\partial\mathbf{x}/\partial t^1$ and $\partial\mathbf{x}/\partial t^2$ span a parallelogram of area $\| (\partial\mathbf{x}/\partial t^1)\times(\partial\mathbf{x}/\partial t^2)\| = \|\mathbf{n}\|$, we have the area element "dA" $= \|\mathbf{n}\|\, dt^1 dt^2$. Our integral (25) then becomes

$$\int_V \beta^2 = \iint_S <\mathbf{B},\ \mathbf{n}> dt^1\, dt^2 = \iint_S \|\mathbf{B}\|\, \|\mathbf{n}\| \cos\angle(\mathbf{B},\ \mathbf{n})\, dt^1\, dt^2$$
$$= \text{classically} \quad \iint_S B_{normal}\, dA$$

and this shows further that the integral $\int_V \beta$ is in fact independent of the parameterization F used.

Note again that our form version (25) **requires no metric or area element**.

Moral. The integrand in a surface integral is naturally a 2–form, not a vector.

One integrates exterior p–forms over oriented p dimensional hypersurfaces V^p. If V^p is not a "closed" hypersurface it will generically have a (p–1) dimensional oriented boundary, written ∂V. For example, if V^2 is oriented, then the circular orientation arrow near the boundary curve of V will yield a "direction" for ∂V (see the surface integral picture, p. 16.)

"**Stokes' Theorem**" $\qquad \int_V d\beta^{p-1} = \oint_{\partial V} \beta^{p-1} \qquad$ (26)

perhaps the World's Most Beautiful Formula. The vector analysis version includes not only Stokes' theorem (really due to **William Thomson= Lord Kelvin**) when p=2 and V^2 is an oriented surface and ∂V is its closed curve boundary, but also **Gauss**' divergence theorem when p=3, V^3 is a bounded region in space and ∂V is its closed surface boundary.

11) **Electromagnetism, or, Is it a Vector or a Form**? For simplicity we consider electric and magnetic fields caused by charges, currents, and magnets in a vacuum (without polarizations, ..)

Electric Field Intensity E. The work done in moving a particle with charge q along a curve C is classically $W = \int_C q\mathbf{E}\bullet d\mathbf{r}$ but really $= q\int_C \mathcal{E}^1 = q\int_C E_1\, dx^1 + E_2 dx^2 + E_3 dx^3$. The electric field intensity is a 1–form $\mathcal{E}^1 = E_1\, dx^1 + E_2 dx^2 + E_3 dx^3$.

Electric Field D. The charge Q contained in a region V^3 with boundary ∂V is classically given by $4\pi\, Q(V^3) = \iint_{\partial V} \mathbf{D}\bullet d\mathbf{A} = \iiint_V \text{div}\, \mathbf{D}\ \text{vol}$, but really $\iint_{\partial V} \mathcal{D}^2 = \iiint_V d\mathcal{D} =$

$4\pi\, Q(V^3) = 4\pi \iiint_V \rho\ \text{vol}^3$, where ρ is charge density. Stokes' theorem thus yields **Gauss' law**

$\boxed{d\,\mathcal{D}^2 = 4\pi\, \rho\ \text{vol}^3}$. $\mathcal{D}^2$ is really a 2–form version of $\mathcal{E}^1$. In **cartesian** coords

$$\mathcal{D}^2 = E_1\, dx^2 \wedge dx^3 + E_2\, dx^3 \wedge dx^1 + E_3\, dx^1 \wedge dx^2.$$

Magnetic Field Intensity B. Faraday's law says classically, for a **fixed** surface V^2, $\oint_{\partial V} \mathbf{E}\bullet d\mathbf{r} = -d/dt \iint_V \mathbf{B}\bullet d\mathbf{A}$. Really $\oint_{\partial V} \mathcal{E}^1 = -d/dt \iint_V \mathcal{B}^2$. Magnetic field intensity is a 2–form $\mathcal{B}^2$ and **Faraday's law** says $\boxed{d\,\mathcal{E}^1 = -\,\partial\mathcal{B}^2/\partial t}$, where $\partial\, \mathcal{B}^2/\partial t$ means take the time derivative of the components of $\mathcal{B}^2$. Another axiom states that $\boxed{\text{div}\ \mathbf{B} = 0 = d\, \mathcal{B}^2}$

Magnetic Field H. Ampere–Maxwell says classically

$\oint_{C=\partial V}$ **H**•d**r**= $4\pi\iint_V$**j**•d**A**+ $d/dt \iint_V$ **D**•d**A** where V^2 is **fixed** and **j** is the current vector. Really

$$\oint_{C=\partial V} \mathcal{H}^1 = 4\pi\iint_V j^2 + d/dt \iint_V \mathcal{D}^2$$

and thus $\boxed{d\,\mathcal{H}^1 = 4\pi\, j^2 + \partial\mathcal{D}^2/\partial t}$, where j^2 = the current 2–form whose integral over V^2 (with a preferred normal direction), measures the time rate of charge passing through V^2 in that direction. $\mathcal{H}^1$ is a 1–form version of $\mathcal{B}^2$. In **cartesian** coords

$$\mathcal{H}^1 = B_{23}dx^1 + B_{31}dx^2 + B_{12}dx^3.$$

Heaviside–Lorentz Force. Classically the electromagnetic force acting on a particle of charge q moving with velocity **v** is given by **f** = q(**E** + **v**×**B**). We have seen that force and the electric field should be 1–forms, $f^1 = q(\mathcal{E}^1 + ??)$. **v** is definitely a vector, and $\mathcal{B}$ is a 2–form! We now discuss this dilemma raised by the vector product × and its resolution will play a large role in our discussion of elasticity also.

12) Interior Products. We are at home with the fact $\alpha^1\wedge\beta^1$ is a 2–form replacement for a × product of vectors in $\mathbb{R}^3$, but if we had started out with two vectors **A** and **B** it would require a metric to change them to 1–forms. It turns out there is also a 1–form replacement that is frequently more useful, and will resolve the Lorentz force problem.

In $\mathbb{R}^n$, if **v** is a vector and β^p is a p–form, p>0, we define the **interior product** of **v** and β to be the (p−1)–form $i_{\mathbf{v}}\beta$ (sometimes we write $i(\mathbf{v})\,\beta$) with values

$$\boxed{i_{\mathbf{v}}\beta^p(\mathbf{A}_2, ,, \mathbf{A}_p) := \beta^p(\mathbf{v}, \mathbf{A}_2, ,, \mathbf{A}_p)} \qquad (27)$$

(It can be shown that this is a contraction, $(i_{\mathbf{v}}\beta)_{bc...} = v^i \beta_{ibc...}$)

This *is* a form since it clearly is multilinear in $\mathbf{A}_2, .. \mathbf{A}_p$, since β is, and changes sign under each interchange of the A's, and is defined independent of any coords. In the case of a 1–form β, $i_{\mathbf{v}}\beta$ is the 0–form (function)

$i_{\mathbf{v}}\beta^1 = \beta^1(\mathbf{v})$ = the function $b_i\, v^i$ which = $<\mathbf{v}, \mathbf{b}>$ in any Riemannian metric.

Look at $i_{\mathbf{v}}(\alpha^1\wedge\beta^1)$. $i_{\mathbf{v}}(\alpha^1\wedge\beta^1)(\mathbf{C}) = (\alpha^1\wedge\beta^1)(\mathbf{v},\mathbf{C}) = \alpha(\mathbf{v})\beta(\mathbf{C}) - \alpha(\mathbf{C})\beta(\mathbf{v})$

$$= (i_{\mathbf{v}}\alpha)\,\beta(\mathbf{C}) - [\alpha\;\; i_{\mathbf{v}}\beta](\mathbf{C}) = [(i_{\mathbf{v}}\alpha)\,\beta - (i_{\mathbf{v}}\beta)\alpha](\mathbf{C})$$

A more gruesome calculation shows the general product rule

$$\boxed{i_{\mathbf{v}}(\alpha^p\wedge\beta^q) = [i_{\mathbf{v}}(\alpha^p)]\wedge\beta^q + (-1)^p\alpha^p\wedge[i_{\mathbf{v}}\beta^q]} \qquad (28)$$

just as for the differential d.

13)**Volume Forms and Cartan's Vector Valued Exterior Forms.** Let x,y be positively oriented Cartesian coords in $\mathbb{R}^2$. The area 2–form in the cartesian plane is $vol^2 = dx\wedge dy$, but in polar coords we have $vol^2 = r\, dr\wedge d\theta$. Looking at (7) we note that $r = \sqrt{g}$, where

$$g := \det(g_{ij}) \tag{29}$$

In any Riemannian metric, in any oriented $\mathbb{R}^n$, we define the volume n–form to be

$$vol^n := \sqrt{g}\, dx^1\wedge\ldots\wedge dx^n \tag{30}$$

in any positively oriented curvilinear coords. It can be shown that this is indeed an n–form (modulo some question of orientation that I do not wish to consider here.) In spherical coords in $\mathbb{R}^3$ we get, since $(g_{ij}) = \mathrm{diag}(1, r^2, r^2\sin^2\theta)$ the familiar $vol^3 = r^2 \sin\theta\, dr\wedge d\theta\wedge d\phi$.

Note now the following in $\mathbb{R}^3$ in any coords. For any vector **v**

$$i_{\mathbf{v}}\, vol^3 = i_{\mathbf{v}} \sqrt{g}\, dx^1\wedge dx^2\wedge dx^3 = \sqrt{g}\; i_{\mathbf{v}}(dx^1\wedge dx^2\wedge dx^3)$$

Now apply the product rule (28) repeatedly

$$i_{\mathbf{v}}(dx^1\wedge dx^2\wedge dx^3) = v^1 dx^2\wedge dx^3 - dx^1 \wedge i_{\mathbf{v}}(dx^2\wedge dx^3) = v^1 dx^2\wedge dx^3 - dx^1 \wedge [v^2 dx^3 - v^3 dx^2]$$
$$= v^1 dx^2\wedge dx^3 - v^2 dx^1\wedge dx^3 + v^3 dx^1\wedge dx^2$$

and so

$$\boxed{\begin{aligned} i_{\mathbf{v}}\, vol^3 &= \sqrt{g}\,[v^1 dx^2\wedge dx^3 - v^2 dx^1\wedge dx^3 + v^3 dx^1\wedge dx^2] \\ &= \sqrt{g}\,[v^1 dx^2\wedge dx^3 + v^2 dx^3\wedge dx^1 + v^3 dx^1\wedge dx^2] \end{aligned}} \tag{31}$$

Remark. For a surface V^2 in Riemannian $\mathbb{R}^3$, with unit normal vector field **n**, it is easy to see that $i_{\mathbf{n}}\, vol^3$ **is the area 2–form** for V^2. Simply look at its value on a pair of vectors **A**, **B** tangent to V; $i_{\mathbf{n}}\, vol^3(\mathbf{A},\mathbf{B}) = vol^3(\mathbf{n}, \mathbf{A},\mathbf{B})$= area spanned by **A** and B.

Comparing (31) with (14) we see that the most general 2–form ν^2 in $\mathbb{R}^3$, in any coords, is of the form $i_{\mathbf{v}}\, vol^3$ where $v^1 = \nu_{23}/\sqrt{g}$, etc. In electromagnetism, $\mathcal{D}^2 = i_{\mathbf{E}} vol^3$. **The same procedure works for an (n–1) form in** $\mathbb{R}^n$. Note that this does not require an entire metric tensor; **we use only the volume element**. If we have a distinguished volume form (i.e., if we have the notion of the volume spanned by a "positively oriented" n–tuple of vectors in $\mathbb{R}^n$), even if it is not derived from a metric, **we shall use the same notation** in positively oriented coords where $\sqrt{g}$ is now merely a fixed given coefficient. (**Warning**, this notation is not standard.)

$$vol^n = \sqrt{g}\, dx^1\wedge\ldots\wedge dx^n$$

If we have a volume form, we can define the **divergence of a vector field v** as follows

$$(\mathrm{div}\, \mathbf{v})\, vol^n := d\, i_{\mathbf{v}}\, vol^n = d \sqrt{g}\,[v^1 dx^2\wedge dx^3\wedge\ldots\wedge dx^n - v^2 dx^1\wedge dx^3 \ldots dx^n + \ldots]$$
$$= [\partial(v^1\sqrt{g})/\partial x^1 + \partial(v^2\sqrt{g})/\partial x^2 + \ldots]\, dx^1\wedge\ldots\wedge dx^n$$

i.e.,

$$\boxed{\mathrm{div}\, \mathbf{v} = (1/\sqrt{g})\, \partial/\partial x^i\,(\sqrt{g}\, v^i)} \tag{32}$$

If, furthermore, the volume form comes from a Riemannian metric we can define the **Laplacian** of a function f by

$$\nabla^2 f := \Delta f := \operatorname{div} \nabla f = (1/\sqrt{g})\, \partial/\partial x^i (\sqrt{g}\, g^{ij}\, \partial f/\partial x^j)) \tag{33}$$

We have seen in section 7) that in $\mathbb{R}^3$ the 2–form

$$\alpha^1 \wedge \gamma^1 = (a_1 dx^1 + a_2 dx^2 + a_3 dx^3) \wedge (c_1 dx^1 + c_2 dx^2 + c_3 dx^3) =$$
$$= (a_2 c_3 - a_3 c_2)\, dx^2 \wedge dx^3 + (a_3 c_1 - a_1 c_3)\, dx^3 \wedge dx^1 + (a_1 c_2 - a_2 c_1)\, dx^2 \wedge dx^3$$

corresponds to the cross product $\mathbf{a}\times\mathbf{c}$ in **cartesian** coords, and this is ideal when considering surface integrals in any coords. We shall now give a 1–form version that will work in any coords and be more useful for considering electromagnetism and especially elasticity in our later sections. In $\mathbb{R}^3$ **with a vol**^{3}, and in any coords, we define

$(\mathbf{a}\times\mathbf{b})^{(1)}$ is the unique **1–form** defined by $(\mathbf{a}\times\mathbf{b})^{(1)}(\mathbf{c}) = \mathrm{vol}^3(\mathbf{a},\mathbf{b},\mathbf{c})$ for every vector $\mathbf{c}$.

If we have a metric, the usual definition of the *vector* $\mathbf{a}\times\mathbf{b}$ is from $\mathbf{a}\times\mathbf{b}\bullet\mathbf{c} = \mathrm{vol}^3(\mathbf{a},\mathbf{b},\mathbf{c})$, but the form version is clearly more general. (Question. How would you define the $\times$ product of n–1 vectors in an $\mathbb{R}^n$ with a vol^n?)

Note

$$\mathrm{vol}^3(\mathbf{a},\mathbf{b},\mathbf{c}) = -\,\mathrm{vol}^3(\mathbf{b},\mathbf{a},\mathbf{c}) = (-i_{\mathbf{b}}\ \mathrm{vol}^3)(\mathbf{a},\mathbf{c}) = -\beta^2(\mathbf{a},\mathbf{c}) = -i_{\mathbf{a}}\, \beta^2(\mathbf{c})$$

where, from (27), $\beta^2 = i_{\mathbf{b}}\mathrm{vol}$ is the 2–form version of $\mathbf{b}$.

$$\boxed{(\mathbf{a}\times\mathbf{b})^{(1)} = -i_{\mathbf{a}}\, \beta^2} \tag{34}$$

Now we can write the Lorentz force law of section 11)

$$\boxed{f^1 = q(\mathcal{E}^1 - i_{\mathbf{v}}\, \mathcal{B}^2)}$$

Finally, an important restatement of the cross product in $\mathbb{R}^3$. We are going to **follow Elie Cartan and use 2–forms whose values on pairs of vectors are not numbers but rather vectors or covectors**. Let χ^2 be the **covector–valued 2–form** with value the covector $\chi(\mathbf{a},\mathbf{b}) := (\mathbf{a}\times\mathbf{b})^{(1)}$. The j^{th} component of this covector is

$$\chi(\mathbf{a},\mathbf{b})_j = (\mathbf{a}\times\mathbf{b})_j = (\mathbf{a}\times\mathbf{b})^{(1)}(\partial_j) = \mathrm{vol}^3(\partial_j, \mathbf{a},\mathbf{b}) = i(\partial_j)\, \mathrm{vol}^3(\mathbf{a},\mathbf{b})$$

Thus

$$\chi^2 = dx^j \otimes \chi_j = dx^j \otimes i(\partial_j)\, \mathrm{vol}^3$$

Note the $\otimes$, **not** $\wedge$. Thus $\chi^2(\mathbf{a},\mathbf{b}) = [\mathrm{vol}^3(\partial_j, \mathbf{a},\mathbf{b})]\, dx^j$. With a Riemannian metric, the **contravariant** version is

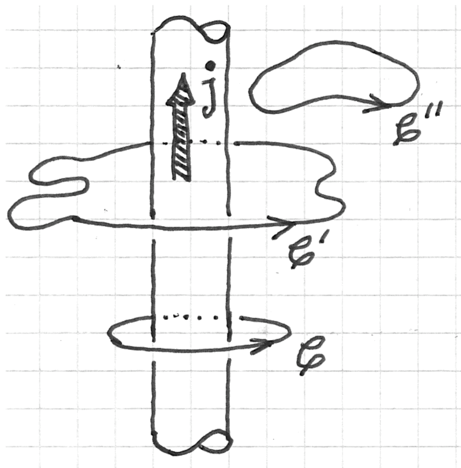
$$\boxed{\chi^2 = \partial_i \otimes g^{ij}\, i(\partial_j)\, \mathrm{vol}^3} \tag{35}$$

This is the vector–valued 2–form that, when applied to a pair of vectors, yields their $\times$ product. In Cartesian coords we can write it symbolically as the column of 2–forms

$$\boxed{[dy\wedge dz \quad dz\wedge dx \quad dx\wedge dy]^T}$$

whose value on a pair of vectors $(\mathbf{a},\mathbf{b})$ is the column of components of $\mathbf{a}\times\mathbf{b}$.

14) Magnetic Field for Current in a straight Wire. This simple example illustrates much of what we have done. Consider a steady current j in a thin straight wire.



Since the current is steady we have Ampere's law $\oint_{C=\partial V} \mathcal{H}^1 = 4\pi \iint_V \mathcal{j}^2$. Looking at three surfaces bounded respectively by C, C', and C'' and the flux of current through them, we have

$\oint_C \mathcal{H}^1 = 4\pi j = \oint_{C'} \mathcal{H}^1$, while $\oint_{C''} \mathcal{H}^1 = 0$. Introducing cylindrical coords, we can guess immediately that $\mathcal{H}^1 = 2j\, d\theta$ **in the region outside the wire**, for it has the correct integrals. We **require**, however, that div $\mathbf{B} = 0 = d\,\mathcal{B}^2$. Now $\mathcal{B}^2 = i_{\mathbf{H}} \mathrm{vol}^3$ where $\mathbf{H}$ is the contravariant version of the 1–form $\mathcal{H}$. The metric for cylindrical coords is diag(1, r^2, 1) and $H_\theta = 2j$ is the only nonzero component of our guess $\mathcal{H}^1$, hence $H^\theta = g^{\theta\theta} H_\theta$ (no sum)$= (1/r^2)\, 2j$. Then $\mathcal{B}^2 = i_{\mathbf{H}} \mathrm{vol}^3$ becomes

$$\mathcal{B}^2 = (2j/r^2)\, i(\partial_\theta)\, r\, dr \wedge d\theta \wedge dz = -(2j/r)\, dr \wedge dz = d\,[-2j\,(\ln r)\, dz]$$

Clearly $d\,\mathcal{B} = 0$, as required, and, in fact, $[-2j\,(\ln r)\, dz]$ is a "magnetic **potential**" 1–form α^1 outside the wire, $\mathcal{B}^2 = d\alpha^1$. $\alpha^1 = 2jz/r\; dr$ is another choice.

ELASTICITY and STRESSES

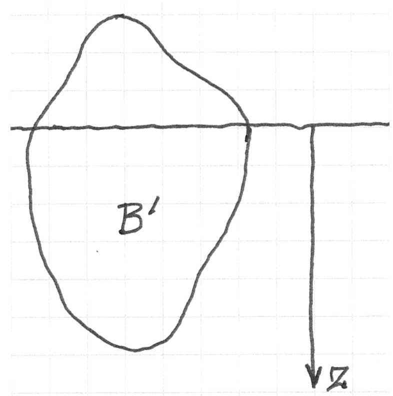
15) **Cauchy stress, floating bodies, twisted cylinders, and stored energy of deformation**. Look at our cylinder B and its twisted version F(B) on page 2, but at **first** we shall use **cartesian** coords x^i. Consider any small surface V in F(B) passing through a point p and let $\mathbf{n}$ be any normal to V at p. Then because of the twisting, the material on the side of V towards which $\mathbf{n}$ is pointing, exerts a force $\mathbf{f}$ on the material on the other side of V. Cauchy's "1st theorem" states that this force is reversed if we replace $\mathbf{n}$ by $-\mathbf{n}$, and further this (contravariant) force is given by integrating a vector valued 2–form $\mathcal{t}$ over V (*not* Cauchy's language)

$$\boxed{\mathbf{f} \text{ on } V = \partial_a \left[\int_V t^{ab}\, i(\partial_b)\, \mathrm{vol}^3\right]}$$

where $\mathbf{t}$ is the "Cauchy stress tensor". A sketch of a proof of Cauchy's theorem will be given in section **16)**. Cauchy's "2nd theorem" says $t^{ab} = t^{ba}$ and a proof sketch is given in section **17)**.

As a warm–up check of our machinery, let us look first at an example of the simplest type of stress from elementary physics. In the case of a **non–viscous fluid**, given a very small parallelogram spanned by **v** and **w** and normal n=v×**w**, the fluid on the side to which **n** is pointing exerts a force on the other side approximated by – p v×**w**, where p is the hydrostatic **pressure**. From (35) the stress vector valued 2–form is given by $\mathcal{t} = -\partial_i \otimes p\, g^{ij}\, i(\partial_j)\, vol^3$.

In a pool with cartesian coords x, y, z, with origin at the surface and z pointing down, look at a floating body B, with portion B′ below the water surface, with surface normal pointing out of B. While Archimedes knew the result, *we* need to practice with our new tools.



Then total stress force exerted on ∂B from water of *constant* density ρ outside B is, with $g^{ij} = \delta^{ij}$,

$$\mathbf{f} = \partial_i \int_{\partial B'} t^{ij}\, i(\partial_j)\, vol^3 = -\partial_i \int_{\partial B'} p\delta^{ij}\, i(\partial_j)\, vol^3$$
$$= -\,\partial_x \int_{\partial B'} \rho gz\, dy\wedge dz - \partial_y \int_{\partial B'} \rho gz\, dz\wedge dx \;-\; \partial_z \int_{\partial B'} \rho gz\, dx\wedge dy$$

where we have included the part of ∂B′ at water level z=0, even though there is no water there, since ρgz= 0 there and we get a 0 contribution from it. We shall evaluate the surface integrals by applying Stokes' theorem (26) to B′. The three 2–forms ρgz dy∧dz, etc, apply only to the outside of B' since there is no water inside B′. To apply Stokes' theorem we must **extend** these 2–forms mathematically to the inside of B′, **in any smooth way that we wish**, and we choose the same forms as are given outside B′, with $\rho = \rho_{water}$ again! Then by Stokes

$$\mathbf{f} = -\partial_x \int_{B'} d[\rho gz\, dy\wedge dz] \;\; -\partial_y \int_{B'} d[\rho gz\, dz\wedge dx] \;\; - \partial_z \int_{B'} d[\rho gz\, dx\wedge dy]$$

$$= -\partial_z \int_{B'} \rho g\, dx\wedge dy\wedge dz = -\partial_z W'$$

where W′ is the weight of the water displaced by B′. Equilibrium demands this must equal the weight of the whole body B. Thus a floating body displaces its own weight in water. EUREKA!

Back to our twisted cylinder. Introduce cylindrical coords (X^A)= (R, Θ, Z) for the untwisted cylinder B. Next, introduce an **identical** set of coords (x^a)=(r,ϑ,z) and use the capitalized coords for a point in the untwisted body and r,ϑ,z for the coords of the image point under the twist F. Thus F is described by r=R, ϑ=Θ+kZ, and z=Z, where k is a constant.We need to determine the **Cauchy** vector valued stress 2–form $\mathcal{t} = \partial_a \otimes \mathcal{t}^{(2)a} = \partial_a \otimes t^{ab}\, i(\partial_b)\, vol^3$

on F(B) in terms of the twisting forces and the material from which B is made. We shall do this by first pulling this 2–form back to the untwisted body B by the following procedure; we **pull** back the 2–forms $\ell^{(2)a}$ by F* and we **push** the vectors ∂_a back to B by the inverse $(F^{-1})_*$, which exists since F is a 1–1 deformation. The resulting vector valued 2–form on B is of the form

$$\boxed{\mathbf{S} = (F^{-1})_*(\partial_a) \otimes F^*\ell^{(2)a} = (F^{-1})_*(\partial_a) \otimes F^* [\, t^{ab}\, i(\partial_b)\, vol^3]}$$

$$\boxed{\mathbf{S} = \partial_A \otimes S^{(2)A} = \partial_A \otimes S^{AB}\, i(\partial_B)\, VOL^3} \qquad (36)$$

called the "**2^nd Piola–Kirchhoff** " vector valued stress 2–form. We shall relate **this** form to the twist F by a generalization of Hooke's law.

We need to know how this twist F has stretched, sheared, rotated, ... the body, and this is described as follows. The euclidean metric is $dS^2 = (dR^2+R^2d\Theta^2+ dZ^2) = ds^2= (dr^2+r^2d\vartheta^2 +dz^2)$. The pull back (section **9**), last paragarph) of ds^2 under the twist F is given by the chain rule

$$\begin{aligned} F^* ds^2 &= F^* (dr^2+r^2d\vartheta^2 +dz^2) = dR^2 +R^2[\, (\partial\vartheta/\partial\Theta)d\Theta+ (\partial\vartheta/\partial Z)dZ\,]^2 +dZ^2 \\ &= dR^2 +R^2[d\Theta+ k dZ]^2 +dZ^2 = dR^2+ R^2[d\Theta^2 +2k\, d\Theta dZ +k^2dZ^2] +dZ^2 \end{aligned}$$

Recall what this is saying. At a point R, Θ,Z of the untwisted body, given two vectors **A**,**B**, we have not only the scalar product $<\mathbf{A}, \mathbf{B}> = dS^2(\mathbf{A},\mathbf{B})$ but also the scalar product of the images after the twist, i.e., from (24), $ds^2(F_*\mathbf{A},F_*\mathbf{B}) =: (F^*ds^2)(\mathbf{A},\mathbf{B})$. Then **one** measure of how much the twist F is distorting distances and angles is defined by the

$$\boxed{\textbf{Lagrange deformation tensor}\ E := (½)[F^* ds^2 - dS^2]} \qquad (37)$$

The quadratic form (covariant second rank tensor) E is determined by its square matrix.

How do the stresses depend on the deformations? In our twisting case we have $E= kR^2d\Theta dZ + ½\, k^2R^2dZ^2$. We will work only to the **first approximation for small k**, i.e., we shall put $k^2 =0$, so $E=kR^2\, d\Theta dZ = ½\, kR^2\, (d\Theta dZ + dZ\, d\Theta)$. We write the components as a symmetric matrix

$$(E_{IJ}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & kR^2/2 \\ 0 & kR^2/2 & 0 \end{bmatrix}$$

The mixed version, using $E^A{}_B = G^{AI}E_{IB}$ and (G^{KL})=diag(1, 1/R^2, 1), is the (non–symmetric)

$$(E^A{}_B) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k/2 \\ 0 & kR^2/2 & 0 \end{bmatrix}$$

and thus tr E = $E^A{}_A$=0 "mod k^2", i.e., putting k^2=0. Finally, putting $E^{AB} = E^A{}_I\, G^{IB}$

$$(E^{AB}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k/2 \\ 0 & k/2 & 0 \end{bmatrix}$$

"**Linear** elasticity" assumes a linear, vastly generalized "Hookes law" relating the stress S to the deformation E. Assuming the body is "**isotropic**" (i.e., the material has no special internal directional structure such as grains in wood), it can then be shown (e.g., [TF, equation (D.9)] that there are then only 2 "elastic constants" μ and λ relating S to E

$$S^{AB} = 2\mu\, E^{AB} + \lambda\, (\mathrm{tr}\, E)\, G^{AB} \qquad (38)$$

and so

$$(S^{AB}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu k \\ 0 & \mu k & 0 \end{bmatrix}$$

This gives rise to the 2nd Piola–Kirchhoff vector valued stress 2–form on the **undeformed** body

$$\mathbf{S}^2 := \partial_I \otimes S^{IJ}\, i(\partial_J)\, \mathrm{vol}^3 = \partial_I \otimes S^{IJ}\, i(\partial_J)\, R\, dR \wedge d\,\Theta \wedge dZ$$

$$= [\partial_\theta \otimes S^{\theta Z}\, i(\partial_Z) + \partial_Z \otimes\ S^{Z\theta}\, i(\partial_\theta)]\, R\, dR \wedge d\,\Theta \wedge dZ$$

$$\boxed{\mathbf{S}^2 = \mu k R\, [\partial_\theta \otimes dR \wedge d\,\Theta + \partial_Z \otimes dZ \wedge dR} \qquad (38')$$

Finally, the **Cauchy stress vector valued 2–form** t on the "current" deformed body

from (36), is $t^2 = F_* \,\partial_A \otimes (F^{-1})^* S^{(2)A}$. Using F^{-1} defined by R=r, $\Theta = \vartheta - kz$, Z=z, we get

$$t^2 = \mu k r\, [\partial_\vartheta \otimes (F^{-1})^*(\, dR \wedge d\,\Theta) + \partial_z \otimes (F^{-1})^*(dZ \wedge dR)\ ]$$

$$= \mu k r\, [\partial_\vartheta \otimes dr \wedge (d\vartheta - kdz) + \partial_z \otimes dz \wedge dr]\ , \qquad \text{and discarding } k^2$$

$$\mathcal{t}^2 = \mu kr\, [\partial_\vartheta \otimes dr\wedge d\vartheta + \partial_z \otimes dz\wedge dr] \qquad (39)$$

To get correct "dimensions" for force we use the "**physical**" components of force, i.e., we normalize the basis vectors. Since $g_{rr} = 1 = g_{zz}$, ∂_r and ∂_z are unit vectors, call them $\mathbf{e}_r$ and $\mathbf{e}_z$. But $g_{\vartheta\vartheta} = r^2$, and so ∂_ϑ, by (6), has length r, and so we put $\mathbf{e}_\vartheta = r^{-1}\, \partial_\vartheta$. ***We make no changes to the form parts dr, dϑ, and dz.***

$$\boxed{\mathcal{t}^2 = \mu kr^2\, \mathbf{e}_\vartheta \otimes dr\wedge d\vartheta + \mu kr\, \mathbf{e}_z \otimes dz\wedge dr} \qquad (40)$$

We shall now see the consequences of this Cauchy stress. Look first at the lateral surface r=a. Then dr=0 there and so $\mathcal{t}$=0 on this surface. **This means that no external "traction" on *this* part of the boundary is needed for this twisting.**

Now look at the end boundary at z=L. From (40) we have stress from outside

$$\mu kr^2\, \mathbf{e}_\vartheta \otimes dr\wedge d\vartheta$$

acting in the $\mathbf{e}_\vartheta$ direction. This has to be supplied by **external tractions** since there is no part of the body past its end. What is the **moment** of the traction? We have a disc, radius a, a force of magnitude $\mu kr^2 dr\, d\vartheta$ acting in the $\mathbf{e}_\vartheta$ direction on an infinitesimal "rectangle" of "sides" dr and dϑ,. The moment about the z axis is $r(\mu kr^2) dr d\vartheta$, and so the total moment is $\mu k \iint r^3 dr d\vartheta = \mu k(a^4/4)2\pi = \pi\mu k a^4/2$. If the total twist at z=L is angle α, $\alpha = kL$, then the total moment required is $\pi\mu a^4\alpha/2L$. An opposite moment is required at z=0. An experiment could yield the value of μ.

In the case of the floating body, treated near the beginning of our section 15), our argument *really* showed the following. Take any blob of fluid B'' surrounded by fluid at rest under the surface z=0. Then the hydrostatic stress (pressure) on $\partial B''$ due to the water surrounding B'', produced a "body force" that supported the weight of the water in B''. We now show that in the case of our twisted cylinder, to order k,

The Cauchy stresses produce no internal **body** forces inside the cylinder

Look at an internal portion B of the cylinder, with boundary ∂B. The Cauchy stress acting on B from outside B derives from the vector valued 2–form in (40) at points of ∂B. For **total** stress force on ∂B, **we cannot just integrate this because it makes no sense to add vectors like $\mathbf{e}_\vartheta$ at different points**. There's no problem with the $\mathbf{e}_z$ components because $\mathbf{e}_z$ is a constant vector field in $\mathbb{R}^3$. So let us express the unit vector $\mathbf{e}_\vartheta$ in terms of the constant basis $\mathbf{e}_x$ and $\mathbf{e}_y$. **Again we leave the cylindrical coord 2–forms alone**. Now

$$\partial/\partial\vartheta = (\partial x/\partial\vartheta)\partial/\partial x + (\partial y/\partial\vartheta)\partial/\partial y = (-r \sin\vartheta)\, \mathbf{e}_x + (r \cos\vartheta)\, \mathbf{e}_y$$

and $\mathbf{e}_\vartheta = r^{-1}(\partial/\partial\vartheta) = -\mathbf{e}_x \sin\vartheta + \mathbf{e}_y \cos\vartheta$, and so (40) becomes

$$\mathcal{t}^2 = \mu k r^2(-\mathbf{e}_x \sin\vartheta + \mathbf{e}_y \cos\vartheta) \otimes dr\wedge d\vartheta + \mu k r\, \mathbf{e}_z \otimes dz\wedge dr.$$

Then, with constant basis, $\iint_{\partial B} \mathbf{e}_x \mu k r^2 \sin\vartheta\, dr\wedge d\vartheta = \mathbf{e}_x \iint_{\partial B} \mu k r^2 \sin\vartheta\, dr\wedge d\vartheta$, etc., and so

$$\iint_{\partial B} \mathcal{t}^2 = -\mathbf{e}_x \iint_{\partial B} \mu k r^2 \sin\vartheta\, dr\wedge d\vartheta + \mathbf{e}_y \iint_{\partial B} \mu k r^2 \cos\vartheta\, dr\wedge d\vartheta + \mathbf{e}_z \iint_{\partial B} \mu k r\, dz\wedge dr.$$

But each integral vanishes; e.g., $\mathbf{e}_x \iint_{\partial B} \mu k r^2 \sin\vartheta\, dr\wedge d\vartheta = \mathbf{e}_x \iiint_B d[\mu k r^2 \sin\vartheta]\wedge dr\wedge d\vartheta = 0$, as desired.

It is a fact, alas, that this simple approach will not work to higher order, keeping terms of order k^2. One can not realize such a simple twist; other deformations are required. See F. D. Murnaghan in the References.

I'd like to emphasize one point brought out in the calculation above. When **integrating vector valued** exterior forms, such as Cauchy's $\partial_i \otimes t^{ij}\, i(\partial_j)\, vol^3$, we were forced to make a change to a constant basis for the vector part, $\partial_i = \mathbf{e}_a A^a{}_i$, **but kept the cylindrical exterior forms**, yielding

$$\iint_{\partial B} \mathbf{e}_a \otimes A^a{}_i t^{ij}\, i(\partial_j)\, vol^3 = \mathbf{e}_a \iint_{\partial B} A^a{}_i t^{ij}\, i(\partial_j)\, vol^3 = \mathbf{e}_a \iiint_B d\,[A^a{}_i t^{ij}\, i(\partial_j)\, vol^3]$$

and our exterior differential completely **avoids Christoffel symbols and tensor divergence** of (t^{ij}) in curvilinear coords, that appear in tensor treatments.

Finally, let's compute the **work done by the traction** acting on the face Z=L, moving each point (R, Θ) to the point $(R, \Theta+\alpha)$. Let $0 \le \beta \le \alpha$. The traction force on the small "rectangle" of sides dR, $d\Theta$ at $(R, \Theta+\beta)$ has, from (38'), covariant component $\approx f_\theta\, dR\, d\Theta = g_{\theta\theta}\, \mu k_\beta R\, dR\, d\Theta = \mu k_\beta R^3 dR d\Theta$, where $k_\beta = \beta/L$. The work done in moving this rectangle from $\beta=0$ to $\beta=\alpha$ is $\approx (dR\, d\Theta) \int_0^\alpha (\mu R^3 \beta/L)\, d\beta = (dR\, d\Theta)\, \mu R^3 \alpha^2/2L$. Thus the total work done in the twist of the face is $W = (\mu\alpha^2/2L) \iint R^3\, dR\, d\Theta = \pi\mu a^4 \alpha^2/4L$. In most common materials ("**hyperelastic**"), in particular for our isotropic body, this work yields an "**energy of deformation**" of the same amount W, that is stored in the twisted body. Furthermore, for hyperelastic bodies, this can be computed from an integral over the undeformed body, [TF, sections A.e. and D.a.],

$$W = (1/2) \iiint S^{AB} E_{AB}\, VOL^3$$

and the reader can verify this in our example using E and S given below (37) and (38).

This is one reason for our choice, at the beginning of this section **15)**, of considering stress force as being contravariant, rather than covariant. The metric deformation tensor E is most naturally covariant E_{AB}, demanding that S be contravariant in order to yield a scalar when computing the work done in a deformation. In terms of our vector valued forms

$E^1 = dX^I \otimes E_{IJ}\, dX^J =: dX^I \otimes E^{(1)}{}_I$ is a covector valued 1–form

$\mathbf{S}^2 = \partial_A \otimes S^{AB}\, i(\partial_B)\, VOL^3 =: \partial_A \otimes S^{(2)A}$ is a vector valued 2–form.

We can then form a **new kind of product** $(\wedge)$ of these two, $\mathbf{S}^2\,(\wedge)\,E^1$, by **taking the wedge product of the forms in both and evaluating the covector of** E^1 **on the vector of** $\mathbf{S}^2$

$$\mathbf{S}^2\,(\wedge)\,E^1 := dX^I(\partial_A)\,[\,S^{(2)A} \wedge E^{(1)}{}_I\,] = S^{(2)A} \wedge E^{(1)}{}_A$$

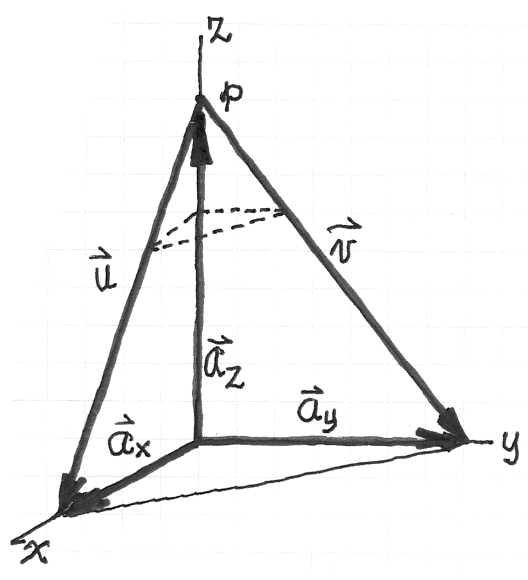
which is easily seen, since the two forms are of complementary dimension, to be the integrand of the stored energy of deformation W

$$\mathbf{S}^2\,(\wedge)\,E^1 = [S^{AB}\, i(\partial_B)\, VOL^3] \wedge E_{AJ}\, dX^J = S^{AB} E_{AB}\, VOL^3$$

$$W = (1/2) \iiint \mathbf{S}^2\,(\wedge)\,E^1$$

While work in particle mechanics pairs a force covector (f_i) with a contravariant tangent vector (dx^i/dt) to a curve, work done by traction in elasticity pairs the contravariant stress force 2–form $\mathbf{S}^2$ with the covector valued deformation 1–form E^1, to yield a scalar valued 3–form. (**Warning.** The notation $(\wedge)$ does not appear in the literature.)

16) **Sketch of Cauchy's "1st Theorem".**



Consider a plane thru a point p on a the z axis of a cartesian coord system. This plane generically cuts the x and y axes at two points, yielding two vectors **u** and **v** that span the "roof" of a solid tetrahedron T, as in the figure above. The coord vectors $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$ are **not** necessarily of the same length. The material outside T exerts a stress force, call it ½ $t(\mathbf{u},\mathbf{v})$ across the roof (½ because the roof is not a parallelogram). $(\mathbf{u},\mathbf{v})$ tells us not only the roof, but also

$\mathbf{u}$, $\mathbf{v}$, in that order is describing the normal pointing out of T. Likewise ½ $t(\mathbf{v},\mathbf{u})$ describes a force that the material in T exerts on material outside T. $t(\mathbf{v},\mathbf{u}) = -\,t(\mathbf{u},\mathbf{v})$ can be seen by considering the equilibrium of a small thin disk with faces parallel to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$. This is the first part of Cauchy's 1st theorem.

Stress forces act also on the coordinate faces. We now let the tetrahedron T shrink to the point p by moving the xy plane up to the point p, the dashed triangle showing an intermediate poition for the bottom face. At each stage the proportions of T are preserved. As the vertical edge $\|\mathbf{a}_z\|$ shrinks to 0, the stresses on the faces vanish as their areas, i.e., as $\|\mathbf{a}_z\|^2$ while the body forces, e.g., gravity, if present, vanish as the volume $\|\mathbf{a}_z\|^3$. We will **neglect the body forces** for vanishingly small T.

For our small T to be in equilibrium we must have, neglecting body forces

$$t(\mathbf{u},\mathbf{v}) + t(\mathbf{a}_z,\mathbf{a}_y) + t(\mathbf{a}_x,\mathbf{a}_z) + t(\mathbf{a}_y,\mathbf{a}_x) \approx 0, \text{ and so}$$

$$t(\mathbf{u},\mathbf{v}) \approx -\,t(\mathbf{a}_z,\mathbf{a}_y) - t(\mathbf{a}_x,\mathbf{a}_z) - t(\mathbf{a}_y,\mathbf{a}_x) \qquad \text{i.e.,}$$

$$t(\mathbf{u},\mathbf{v}) \approx t(\mathbf{a}_y,\mathbf{a}_z) + t(\mathbf{a}_z,\mathbf{a}_x) + t(\mathbf{a}_x,\mathbf{a}_y) \qquad (41)$$

Look at the first term $t(\mathbf{a}_y,\mathbf{a}_z)$. The normal to the pair $\mathbf{a}_y,\mathbf{a}_z$ is in the positive x direction and so the area form for the y,z face is $dy\wedge dz$. Let $<t_{yz}>$ be the **area vector average** of the vector $t(\mathbf{a}_y,\mathbf{a}_z)$, so

$$t(\mathbf{a}_y,\mathbf{a}_z) = <t_{yz}>\ dy\wedge dz\,(\mathbf{a}_y,\mathbf{a}_z).$$

Now note that for projected areas, $dy\wedge dz\,(\mathbf{u},\mathbf{v}) = dy\wedge dz\,(-\mathbf{a}_z,\ -\mathbf{a}_z + \mathbf{a}_y) = dy\wedge dz\,(-\mathbf{a}_z,\ -\mathbf{a}_z) + dy\wedge dz\,(-\mathbf{a}_z,\ \mathbf{a}_y) = dy\wedge dz\,(-\mathbf{a}_z,\ \mathbf{a}_y) = -dy\wedge dz\,(\mathbf{a}_z,\ \mathbf{a}_y) = dy\wedge dz\,(\mathbf{a}_y,\ \mathbf{a}_z)$. Thus

$$dy\wedge dz\,(\mathbf{a}_y,\ \mathbf{a}_z) = dy\wedge dz\,(\mathbf{u},\ \mathbf{v}) \quad \text{and so} \quad t(\mathbf{a}_y,\mathbf{a}_z) = <t_{yz}>\ dy\wedge dz\,(\mathbf{u},\mathbf{v})$$

and similarly for the other faces in (41). We then have

$$t(\mathbf{u},\mathbf{v}) \approx <t_{yz}>\ dy\wedge dz\,(\mathbf{u},\mathbf{v}) + <t_{zx}>\ dz\wedge dx\,(\mathbf{u},\mathbf{v}) + <t_{xy}>\ dx\wedge dy\,(\mathbf{u},\mathbf{v}) \qquad (42)$$

Now as T shrinks to the point p the average $< t_{yz}> \to$ a **vector** $t^x(p) = t^1(p)$ at p, etc.

We can then approximate the stress in (42) , for a very small parallelogram at p spanned by **u** and **v**

$$t(\mathbf{u},\mathbf{v}) \approx [t^x(p)\, dy\wedge dz + t^y(p)\, dz\wedge dx + t^z(p)\, dx\wedge dy\,]\,(\mathbf{u},\mathbf{v})$$

which suggests Cauchy's theorem, that for any surface V^2 with normal direction prescribed, the stress across V is given by a vector valued integral of the form

$$\int_V t^x(x,y,z)\, dy\wedge dz + t^y(x,y,z)\, dz\wedge dx + t^z(x,y,z)\, dx\wedge dy$$

with Cauchy vector valued stress 2–form

$$\boxed{\begin{gathered} t^2 = t^{(2)j} \otimes i(\partial_j)\, vol^3 = \\[2em] t^x(x,y,z)\otimes dy\wedge dz + t^y(x,y,z)\otimes dz\wedge dx + t^z(x,y,z)\otimes dx\wedge dy \end{gathered}}$$

The i^{th} vector *component* of this form can be written $t^{(2)i} = t^{ij}\, i(\partial_j)\, vol^3$. Let us write this out in

classical engineering notation. If **n** is the *unit* normal to a surface element with area "form" dA, then $i(\partial_j)\, vol^3$, when evaluated on vectors tangent to V, is simply $n_j\, dA$, where the components of the normal are $n_j = \cos\angle \mathbf{n}, \partial_j$; this is simply the classical definition of the area form dA. Thus $t^{(2)i} = t^{ij}\, n_j\, dA$, which is the classical expression for Cauchy stress.

17) Sketch of Cauchy's "2^{nd} Theorem". Moments as generators of Rotations.

For Cauchy's 2^{nd} theorem, the symmetry of the stress tensor $t^{ij} = t^{ji}$, we shall consider only the simplest case of a deformed body, at **rest** and in **equilibrium** with its external tractions on its boundary, and with **no external body forces** (like gravity) considered. We employ **Cartesian** coords throughout. Then, since $g_{ij} = \delta_{ij}$, tensorial indices may be raised and lowered indiscriminately and we can use the summation convention for *all* repeated indices.

Let B be any sub body in the **interior** of the body, with boundary ∂B. Then the (assumed vanishing) total stress force covector on B yields

$$0 = \int_{\partial B} \{dx^c\} \otimes t_c{}^b\, i(\partial_b)\, vol^3 = \{dx^c\} \int_{\partial B} t^{(2)}{}_c = \{dx^c\} \int_B dt^{(2)}{}_c\, ,$$

where we use the braces { } just to remind us that the form to the left of $\otimes$ is **a constant covector that plays no role in the integral**. Since this holds for every interior B we must have

$$dt^{(2)}{}_c = d\, t_c{}^b\, i(\partial_b)\, vol^3 = 0 \qquad \text{for each } c \qquad (43)$$

which classically is written as a divergence $\partial t_c{}^b/\partial x^b = 0$.

For equilibrium we must also have that the total **moment** of stress forces on ∂B must vanish. Now the moment about the origin, of a force $\mathbf{f}$ at position vector $\mathbf{r}$ is, in elementary point mechanics, $\mathbf{r}\times\mathbf{f}(\mathbf{r})$, but this expression makes no sense in more than 3 dimensions. But moments and torques surely make sense in any Euclidean $\mathbb{R}^n$, indicating that we have not understood *mathematically* the notion of moment. (See my quote of Hermann Weyl's at the beginning of Appendix A of my text.) Now in Cartesian coords in $\mathbb{R}^n$, if we replace $\mathbf{r}$ and $\mathbf{f}(\mathbf{r})$ by 1−forms $r^1 = x^a\, dx^a$ and $f^1 = f_c(\mathbf{r})\, dx^c$, then $r^1 \wedge f^1$ does make sense as a 2−form **at the origin** of $\mathbb{R}^n$ and its components, in the case of $\mathbb{R}^3$, coincide with those of $\mathbf{r}\times\mathbf{f}(\mathbf{r})$. There is a more important point. A moment about the origin 0 of $\mathbb{R}^n$ is **physically** a "generator" of a rotation about 0. Let us see why a 2−form at the origin of $\mathbb{R}^n$, with components forming a skew symmetric matrix, also is associated to a rotation there.

Let g(t) be a 1−parameter group (i.e., g(t) g(s) = g(t+s), and g(0)=I) of **rotations** of $\mathbb{R}^n$ about the origin. Since each g(t) is an "orthogonal" matrix, $g(t)\, g(t)^T = I$, where T is transpose. Differentiate with respect to t (indicated by an overdot) and put t=0.

$$0 = \dot g(0)\, g(0)^T + g(0) \dot g(0)^T = \dot g(0) + \dot g(0)^T$$

says then that $A := \dot g(0)$ (the so−called "**generator**" of the 1−parameter group g(t)), is a skew symmetric n×n matrix, and so defines a 2−form $A^2 = \Sigma_{j<k} A_{jk}\, dx^j \wedge dx^k$ at the origin. For example, a 1−parameter group of rotations about the z axis of $\mathbb{R}^3$ is, with ω a constant,

$$g(t) = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and has generator} \quad A = \dot g(0) = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with associated 2−form $A^2 = -\omega\, dx\wedge dy$ at the origin. If $\mathbf{v}$ is a vector at the origin, then $A\mathbf{v}$ is the vector $(A\mathbf{v})_j = A_{jk} v^k = -\, v^k A_{kj}$, i.e., the covector version of $A\mathbf{v}$ is $-i(\mathbf{v})\, A^2$.

Conversely, if A is a skew symmetric n×n matrix at the origin, (a 2−form at the origin), then A generates a 1−parameter group of rotations g(t) by means of the exponential matrix

$$g(t) = e^{tA} = \exp tA := \Sigma_k\, t^k A^k / k!$$

(it is an orthogonal matrix since $g(t)^T = \exp tA^T = \exp(-tA) = g(-t) = g^{-1}(t)$.) **A 2−form at the origin of $\mathbb{R}^n$ generates a 1 parameter group of rotations about the origin of $\mathbb{R}^n$.** (Linear algebra also shows that the generator of e^{tA} is $d/dt\, e^{tA}]_{t=0} = A\, e^0 = A$.)

Thus to each moment of a force $\mathbf{f}$ about the origin of $\mathbb{R}^n$ we may attach the generator of its rotations, i,e,. a 2−form at the origin.

Then with our sub body B of an elastic body, the Cauchy stress **covector** valued 2−form yields an "area covector force density" with "components" the 2−forms $F_c = t_c{}^b\, i(\partial_b)\, vol^3$ on the boundary ∂B. The "moment about an origin (chosen inside B)" density has **Cartesian** "components" the 2−forms

$$m_{ac} = [x^a\, t_c{}^b - x^c\, t_a{}^b]\, i(\partial_b)\, vol^3 = x^a\, t^{(2)}{}_c - x^c\, t^{(2)}{}_a$$

Thus the **total moment** about the origin due to these stress forces on ∂B is the 2–form at the origin $\Sigma_{a<c} M_{ac}\, dx^a \wedge dx^c$ with components the numbers

$$M_{ac} = \int_{\partial B} [x^a\, t^{(2)}{}_c - x^c\, t^{(2)}{}_a] = \int_B d\, [x^a\, t^{(2)}{}_c - x^c\, t^{(2)}{}_a]$$

which, from (43), is

$$M_{ac} = \int_B dx^a \wedge t^{(2)}{}_c - dx^c \wedge t^{(2)}{}_a$$

In most common elastic materials, materials with no "micro structure", this must vanish if there are to be no internal rotations without internal torque sources. Since this holds for any portion B we must have

$$dx^a \wedge t^{(2)}{}_c = dx^c \wedge t^{(2)}{}_a \qquad (44)$$

Since these are 3–forms in $\mathbb{R}^3$,

$$dx^a \wedge t^{(2)}{}_c = dx^a \wedge t_c{}^b i(\partial_b)\, vol^3 = t_c{}^a\, vol^3. \qquad (44')$$

For example, in $\mathbb{R}^3$ with a=2 and c =1,

$$dx^2 \wedge t_1{}^b\, i(\partial_b)\, dx^1 \wedge dx^2 \wedge dx^3 = dx^2 \wedge t_1{}^2\, i(\partial_2)\, dx^1 \wedge dx^2 \wedge dx^3 =$$
$$-\, dx^2 \wedge t_1{}^2\, i(\partial_2)\, dx^2 \wedge dx^1 \wedge dx^3 = -\, dx^2 \wedge t_1{}^2 \wedge dx^1 \wedge dx^3 =$$
$$-t_1{}^2\, dx^2 \wedge dx^1 \wedge dx^3 = t_1{}^2\, dx^1 \wedge dx^2 \wedge dx^3 = t_1{}^2\, vol^3$$

(44') then yields $t_c{}^a\, vol^3 = t_a{}^c\, vol^3$, and since coords are cartesian we have

$$t^{ca} = t^{ac}. \qquad (45)$$

Since the Cauchy stress t *is* a tensor, this symmetry holds in **any** coordinate system. This is Cauchy's 2nd Theorem.

Warning. We have previously allowed and encouraged the use of different coordinates for the 2–form part and the value part of the stress vector valued 2–form, see e.g., p. 26

$$\partial_i \otimes t^{ij}\, i(\partial_j)\, vol^3 = \mathbf{e}_a \otimes A^a{}_i\, t^{ij}\, i(\partial_j)\, vol^3 =: \mathbf{e}_a \otimes \tau^{aj}\, i(\partial_j)\, vol^3.$$

The left index "a" on τ is **always** associated with the $\mathbf{e}$ basis and the right index "j" is **always** associated with the ∂ basis. (Think of $\mathbf{e}$ as cartesian and ∂ as cylindrical.) Does the fact that t is symmetric, $t^T = t$, ensure that $\tau = A\, t$ is also ? No!

$$\tau^T = t^T A^T = t\, A^T = (A^{-1}\tau)\, A^T \text{ is generically} \neq \tau!$$

18) A Magic Formula for differentiating Line,and Surface, and ..., Integrals. Let $\mathbf{v}$ be a **time independent** vector field in a coord patch U of $\mathbb{R}^n$ with any coords x^i. Roughly speaking, i.e., omitting some technicalities, by integrating the differential equations $dx^i/dt = v^i(x)$ we move along the integral curves of $\mathbf{v}$ for t seconds yielding a "flow" $\phi_t : U \to \mathbb{R}^n$. Since $\mathbf{v}$ is time independent, the ϕ_t form a 1 parameter commutative group of mappings, $\phi_t\, \phi_h = \phi_{t+h}$ and ϕ_0 is the identity map. Let V^r be an oriented r–dimensional "submanifold" of U. For examples, V^1 is an oriented curve , V^2 is an oriented 2 dimensional surface, ... V^r is the kind of object over which one integrates an exterior r–form $\alpha = \alpha^r$,(a **scalar** valued, **not vector** valued form), yielding the number $\int_V \alpha^r$. As time changes, the flow moves V from V(0) =V to V(t)= ϕ_t (V). We consider only the simplest case where the r−form α is time−independent. How does the integral change in time? The answer can be shown to be

$$d/dt|_{t=0} \int_{V(t)} \alpha^r = \int_V \mathcal{L}_{\mathbf{v}} \alpha^r \qquad (46)$$

where the r−form $\mathcal{L}_{\mathbf{v}}\, \alpha^r$, the **Lie derivative** of the form α, is defined via the pullbacks

$$[\mathcal{L}_{\mathbf{v}} \alpha^r] \text{ (at x) } := d/dt\,]_{t=0}\ \phi_t{}^*[\alpha^r \text{ (at } \phi_t\, x)] \qquad (47)$$

Furthermore, there is a remarkable expression for computing the Lie derivative given by the **Henri Cartan** (son of Elie Cartan) **formula**

$$\boxed{\mathcal{L}_{\mathbf{v}}\, \alpha^r = i_{\mathbf{v}}\, d\alpha^r + d\, i_{\mathbf{v}} \alpha^r \qquad (48)}$$

Thus (46) and Stokes say

$$\boxed{d/dt|_{t=0} \int_{V(t)} \alpha^r = \int_V \mathcal{L}_{\mathbf{v}} \alpha^r = \int_V i_{\mathbf{v}}\, d\alpha + \int_{\partial V} i_{\mathbf{v}}\, \alpha \qquad (49)}$$

Consider for example the case of a line integral in $\mathbb{R}^3$, which we also write in classical form. V^1 is then a curve C starting at point P and ending at point Q. Symbolically $\partial C = Q - P$.

Classically $\alpha = \mathbf{a}\bullet d\mathbf{x}$. Then $i_{\mathbf{v}}\alpha$ is the 0−form, i.e., function $v\bullet\mathbf{a}$, and $\int_{\partial C} \mathbf{v}\bullet\mathbf{a}$ is by **definition** simply $\mathbf{v}\bullet\mathbf{a}(Q) - \mathbf{v}\bullet\mathbf{a}(P)$. This is the second "integral" in (49). Also, $d\alpha^1$ is the 2−form version of the vector curl $\mathbf{a}$, and so $i_{\mathbf{v}}\, d\alpha$, from (34), is the 1−form version of $-\mathbf{v}\times$ curl $\mathbf{a}$. We then have, in the classical version

$$d/dt]_{t=0} \int_{C(t)} \mathbf{a}\bullet d\mathbf{x} = -\int_{C(0)} [\mathbf{v}\times \text{curl } \mathbf{a}] \bullet d\mathbf{x} \;\; + [\mathbf{v}\bullet\mathbf{a}(Q) - \mathbf{v}\bullet\mathbf{a}(P)]$$

The reader should work out the case of the 3−form vol^3 in $\mathbb{R}^3$.

References

T. Frankel, *The Geometry of Physics*, 2nd Ed. Cambridge University Press, 2004
F.D. Murnaghan, *Finite deformation of an elastic solid* , New York, Dover Publications [1967, c1951]