# GROTHENDIECK POLYNOMIALS OF INVERSE FIREWORKS PERMUTATIONS 

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#### Abstract

Pipedreams are combinatorial objects that compute Grothendieck polynomials. We introduce a new combinatorial object that naturally recast the pipedream formula. From this, we obtain the first direct combinatorial formula for the top degree components of Grothendieck polynomials, also known as the Castelnuovo-Mumford polynomials. We also prove the inverse fireworks case of a conjecture of Mészáros, Setiabrata, and St. Dizier on the support of Grothendieck polynomials.


## 1. Introduction

Fix $n \in \mathbb{Z}_{>0}$ throughout the paper. For $w \in S_{n}$, Lascoux and Schützenberger [LS82] introduced the Grothendieck polynomials $\mathfrak{G}_{w}(\mathbf{x})$, which are explicit polynomial representatives of the $K$-classes of structure sheaves of Schubert varieties in flag varieties. In general, the Grothendieck polynomials are not homogeneous. Their lowest degree homogeneous components recover the Schubert polynomials, which represent the cohomology classes of Schubert varieties in flag varieties.

Grothendieck polynomials can be computed using combinatorial objects such as pipedreams (PD) [BB93, BJS93, FK94]. Each $w \in S_{n}$ is associated with a set of PDs, denoted as PD $(w)$. Each $P \in \operatorname{PD}(w)$ is associated with a set of weight tiles wty $(P)$, which leads to a monomial $\mathrm{wt}_{P}(\mathbf{x})$. By Fomin and Kirillov [FK94],

$$
\mathfrak{G}_{w}(\mathbf{x}):=\sum_{P \in \operatorname{PD}(w)}(-1)^{|\mathrm{wty}(P)|-\ell(w)} \mathrm{wt}_{P}(\mathbf{x}) .
$$

The main innovation of this paper is an alternative perspective for viewing PDs. We remove certain pipes from a PD to obtain a novel combinatorial object called a marked vertical-less pipedream (see Definition 3.1). In other words, we introduce a set MVPD $(w)$ and a bijection $\Phi: \operatorname{PD}(w) \rightarrow \operatorname{MVPD}(w)$. We also define the weighty tiles wty $(M)$ of each $M \in \operatorname{MVPD}(w)$ so that $\Phi$ preserves wty $(\cdot)$. Consequently, each $M \in \operatorname{MVPD}(w)$ is associated with a monomial $\mathrm{wt}_{M}(\mathbf{x})$ which agrees with its corresponding PD. This set MVPD $(w)$ recasts the PD formula in a way that helps us derive two applications on Grothendieck polynomials.
1.1. Combinatorial formula for the top degree components of Grothendieck polynomials. There has been a growing acknowledgment of the significance of the CastelnuovoMumford polynomial $\widehat{\mathfrak{G}}_{w}(\mathbf{x})$, the top degree homogeneous component of $\mathfrak{G}_{w}(\mathbf{x})$. For instance,

- the degree of $\widehat{\mathfrak{G}}_{w}(\mathbf{x})$ determines the Castelnuovo-Mumford regularity of matrix Schubert varieties $\left[\mathrm{RRR}^{+} 21\right]$;
- the support of $\widehat{\mathfrak{G}}_{w}(\mathbf{x})$ conjecturally governs the support of $\mathfrak{G}_{w}(\mathbf{x})$ [MSSD22, Conjecture 1.3].

With these motivations，there has been a recent surge in the study of $\widehat{\mathfrak{G}}_{w}(\mathbf{x})$［CY23， DMSD22，Haf22，HMSSD23，PSW21，PY23，RRR ${ }^{+}$21，RRW23］．Any permutation can be decomposed into decreasing runs．A permutation is called fireworks if the first number in each decreasing run is increasing．For instance， 3167542 is fireworks while 6137542 is not． A permutation is called inverse fireworks permutation if its inverse is fireworks．Pechenik， Speyer，and Weigandt［PSW21］showed that each $\widehat{\mathfrak{G}}_{w}(\mathbf{x})$ is an integer multiple of $\widehat{\mathfrak{G}}_{u}(\mathbf{x})$ for some inverse fireworks permutation $u$ ．Thus，to understand all $\widehat{\mathfrak{G}}_{w}(\mathbf{x})$ ，one might focus on $\widehat{\mathfrak{G}}_{u}(\mathbf{x})$ for inverse fireworks $u$ ．

As far as the authors are aware，there does not exist any combinatorial formula for $\widehat{\mathfrak{G}}_{w}(\mathbf{x})$ ， besides extracting highest degree elements from a combinatorial formula for $\mathfrak{G}_{w}(\mathbf{x})$ ．We provide the first direct combinatorial formula of $\widehat{\mathfrak{G}}_{w}(\mathbf{x})$ for inverse fireworks $w$ ，involving combinatorial objects we call bumpless vertical－less pipedreams（BVPDs）．We prove our formula by establishing a bijection between BVPDs and the highest degree elements from $\operatorname{MVPD}(w)$ ．Via the bijection between $\operatorname{MVPD}(w)$ and $\operatorname{PD}(w)$ ，we also obtain a characterization of highest degree PDs of inverse fireworks permutations．

1．2．Support of Grothendieck polynomials for inverse fireworks permutations． The support of a polynomial $f$ is the set of monomials whose coefficient in $f$ is non－zero． Mészáros，Setiabrata，and St．Dizier［MSSD22］made several conjectures on the support for $\mathfrak{G}_{w}(\mathbf{x})$ ，including the following．

Conjecture 1．1．［MSSD22，Conjecture 1．2］Let $w$ be an arbitrary permutation．Let $m$ be an monomial in the support of $\mathfrak{G}_{w}(\boldsymbol{x})$ ．If the degree of $m$ is less than the degree of $\mathfrak{G}_{w}(\boldsymbol{x})$ ， then there exists $i$ such that $m x_{i}$ is in the support of $\mathfrak{G}_{w}(\boldsymbol{x})$ ．

The conjecture above is implied by another conjecture of Huh，Matherne，Mészáros and St．Dizier［HMMSD22］that homogenized Grothendieck polynomials are Lorentzian（up to appropriate normalization）．We may translate Conjecture 1.1 combinatorially：For each $M \in$ $\operatorname{MVPD}(w)$ that is not a top degree element，we may find $M^{\prime} \in \operatorname{MVPD}(w)$ such that $\mathrm{wt}_{M^{\prime}}(\mathbf{x})=$ $\mathrm{wt}_{M}(\mathbf{x}) x_{i}$ for some $i$ ．We prove Conjecture 1.1 for an inverse fireworks permutation $w$ constructively：For each $M \in \operatorname{MVPD}(w)$ ，we give an explicit algorithm that constructs $M^{\prime} \in \mathrm{PD}(w)$ ．

The rest of the paper is structured as follows．In §2，we cover necessary background regarding pipedreams and $\mathfrak{G}_{w}(\mathbf{x})$ ．In $\S 3$ ，we introduce marked vertical－less pipedreams and use them to rephrase the PD formula．In $\S 4$ ，we introduce bumpless vertical－less pipedreams and give a direct formula of $\widehat{\mathfrak{G}}_{w}(\mathbf{x})$ ．In $\S 5$ ，we prove Conjecture 1.1 for inverse fireworks permutations．

## 2．Background

Definition 2．1．A pipedream（PD）［BB93，BJS93，FK94］is a tiling of an $n \times n$ grid．The tile $(i, n+1-i)$ is $\square$ for $i \in[n]$ ．The cell $(i, j)$ with $j>n+1-i$ is $\square$ ．All other tiles can be $\boxtimes$ or $\boxplus$ ．We trace the pipes in a PD from left to top as follows．The pipe entering from row $p$ is called pipe $p$ ．Suppose we see a $⿴ 囗 十$ where the pipe on the left（resp．bottom）has label $p$ （resp．q）．If pipe $p$ and $q$ have not crossed before，we say they cross in this tile and let pipe $p$（resp．$q$ ）exits from the right（resp．top）．Otherwise，we let pipe $p$（resp．$q$ ）exits from the top（resp．right）．Notice that this rule is the same as saying pipe $\max (p, q)$ exits from the top and the other exits from the right．

After tracing the pipes，we may read off the labels of the pipes on the top edge of the PD as a permutation $w \in S_{n}$ ．We say this PD is associated with $w^{-1}$ ．Let $\operatorname{PD}(w)$ be the set of the pipedreams associated with $w$ ．

Example 2．2．The following is a pipedream of the permutation $w$ with one－line notation 24513．Its inverse has one－line notation 41523.


We make pipe 3 blue and pipe 5 green．Notice that pipe 3 and pipe 5 cross at（3，2）．However， pipe 3 and pipe 5 do not cross at $(2,3)$ since they already crossed．

For a PD $P$ ，let wty $(P)$ be the set of $(i, j)$ that is $⿴ 囗 十$ in $P$ ．Define

$$
\mathrm{wt}_{P}(\mathbf{x})=\prod_{(i, j) \in \mathrm{wty}(P)} x_{i}, \quad \mathrm{wt}_{P}(\mathbf{x}, \mathbf{y})=\prod_{(i, j) \in \mathrm{wty}(P)}\left(x_{i}+y_{j}-x_{i} y_{j}\right)
$$

For $w \in S_{n}$ ，let $\ell(w)$ be the number of $i<j$ such that $w(i)>w(j)$ ．Following［FK94］ and［KM05］，Grothendieck polynomial $\mathfrak{G}_{w}(\mathbf{x})$ and double Grothendieck polynomial $\mathfrak{G}_{w}(\mathbf{x}, \mathbf{y})$ can be defined as

$$
\begin{align*}
\mathfrak{G}_{w}(\mathbf{x}) & :=\sum_{P \in \operatorname{PD}(w)}(-1)^{|\operatorname{wty}(P)|-\ell(w)} \mathrm{wt}_{P}(\mathbf{x}) \\
\mathfrak{G}_{w}(\mathbf{x}, \mathbf{y}) & :=\sum_{P \in \operatorname{PD}(w)}(-1)^{|\operatorname{wty}(P)|-\ell(w)} \mathrm{wt}_{P}(\mathbf{x}, \mathbf{y}) . \tag{1}
\end{align*}
$$

Example 2．3．The following are all the pipedreams of the permutation $w=2413$ ：


Therefore，

$$
\begin{aligned}
\mathfrak{G}_{w}(\mathbf{x}) & =x_{1} x_{2}^{2}+x_{1}^{2} x_{2}-x_{1}^{2} x_{2}^{2} \\
\mathfrak{G}_{w}(\mathbf{x}, \mathbf{y}) & =\left(x_{1}+y_{1}-x_{1} y_{1}\right)\left(x_{2}+y_{1}-x_{2} y_{1}\right)\left(x_{2}+y_{2}-x_{2} y_{2}\right) \\
& +\left(x_{1}+y_{1}-x_{1} y_{1}\right)\left(x_{2}+y_{1}-x_{2} y_{1}\right)\left(x_{1}+y_{3}-x_{1} y_{3}\right) \\
& -\left(x_{1}+y_{1}-x_{1} y_{1}\right)\left(x_{2}+y_{1}-x_{2} y_{1}\right)\left(x_{2}+y_{2}-x_{2} y_{2}\right)\left(x_{1}+y_{3}-x_{1} y_{3}\right)
\end{aligned}
$$

We make one simple observation on PDs that will be useful later．
Lemma 2．4．Say three pipes enter a row of a PD from the bottom：Pipe a enters on the left of pipe $b$ and pipe $b$ enters on the left of pipe c．Suppose pipe $a$ and pipe $b$ have not crossed， but pipe $a$ and pipe $c$ have crossed．Then pipe $b$ and pipe $c$ must have crossed．

Proof．Since pipe $a$ enters the row on the left of pipe $b$ and they did not cross，we have $a<b$ ． Since pipe $a$ enters the row on the left of pipe $c$ and they have crossed，we have $c<a$ ．Thus， $c<a<b$ ．Since pipe $b$ enters the row on the left of pipe $c$ ，they have crossed．

The degree of $\mathfrak{G}_{w}(\mathbf{x})$ was given by the statistic $\operatorname{raj}(w)$ defined by Pechenik，Speyer，and Weigandt［PSW21］．In other words，among PDs in $\operatorname{PD}(w)$ ，the maximal number of weighty tiles is $\operatorname{raj}(w)$ ．We define $\widehat{\mathfrak{G}}_{w}(\mathbf{x})$ as $\sum_{P} \mathrm{wt}_{P}(\mathbf{x})$ where the sum is over all $P \in \mathrm{PD}(w)$ with $\operatorname{raj}(w)$ weighty tiles．Up to a sign，$\widehat{\mathfrak{G}}_{w}(\mathbf{x})$ agrees with the top degree component of $\mathfrak{G}_{w}(\mathbf{x})$ ． In this paper，we need the following two properties of the raj statistic．

Proposition 2.5 （［PSW21，Proposition 3．8］）．For $w \in S_{n}, \operatorname{raj}(w)=\operatorname{maj}(w)$ if and only if $w$ is fireworks．Here， $\operatorname{maj}(w)$ is the major index of $w$ ，defined as $\sum_{i: w(i)>w(i+1)} i$ ．
Corollary 2.6 （［PSW21，Corollary 4．5］）．For $w \in S_{n}, \operatorname{raj}(w)=\operatorname{raj}\left(w^{-1}\right)$ ．

## 3．Marked Vertical－Less Pipedreams

We introduce combinatorial objects which we call marked vertical－less pipedreams（MVPD）． An MVPD can be obtained by removing certain pipes from a PD．We rephrase（1）and obtain MVPD formulas for $\mathfrak{G}_{w}(\mathbf{x})$ and $\mathfrak{G}_{w}(\mathbf{x}, \mathbf{y})$ in Corollary 3．7．

Definition 3．1．A vertical－less pipedream（VPD）consists of the following six tiles：

on an $n \times n$ grid．Notice that we are not using the vertical tile $\square$ ．The pipes of a VPD enter from the left edge of the $n \times n$ grid and exit from the top edge．We trace pipes from left to top in the same way as PDs．The pipe entering from row $p$ is called pipe $p$ ．A marked vertical－less pipedreams（MVPD）is a VPD where some $\square$ are marked as $\boldsymbol{\theta}$ ．The pipe in a marked tile must have a $\boxminus$ on the left of this tile．

The column－to－row code of a MVPD $M$ is a sequence of $n$ numbers．If there is no pipe exiting at column $c$ of $M$ ，then the $c^{\mathrm{th}}$ entry is 0 ．Otherwise，say pipe $r$ exits in column $c$ ， then the $c^{\text {th }}$ entry is $r$ ．When drawing a MVPD，we omit blank rows on the bottom and blank columns to the right．

Example 3．2．Suppose $n=8$ ．The following is a MVPD which has column－to－row code $(0,0,0,4,0,3,0,6)$ ．Notice that $(1,2),(2,1)$ ，and $(5,1)$ cannot be marked while $(3,4)$ may or may not be marked．


For a MVPD $M$ ，we let $w t y(M)$ be the set of $(i, j)$ that is $\boxminus$ ，$⿴ 囗 十$ or $\boldsymbol{\theta}$ in $M$ ．We define

$$
\mathrm{wt}_{M}(\mathbf{x})=\prod_{(i, j) \in \mathrm{wty}(M)} x_{i}, \quad \mathrm{wt}_{M}(\mathbf{x}, \mathbf{y})=\prod_{(i, j) \in \mathrm{wty}(M)}\left(x_{i}+y_{j}-x_{i} y_{j}\right)
$$

We associate certain MVPDs to each permutation $w \in S_{n}$ ．The left－to－right maximums of $w \in S_{n}$ are the numbers $w(i)$ such that $w(j)<w(i)$ for all $j<i$ ．For instance，the left－to－right maximums of the permutation with one－line notation 2143 are 2 and 4.

Remark 3．3．Notice that in the one－line notation of a permutation，its left－to－right maximums must increase from left to right．Consequently，in $P \in \mathrm{PD}(w)$ ，pipes labeled by left－to－right maximums of $w^{-1}$ cannot cross．

Take $w \in S_{n}$ ．We start from $\left(w^{-1}(1), \cdots, w^{-1}(n)\right)$ and turn the left－to－right maximums of $w^{-1}$ into 0 ．Let $\alpha^{\prime}(w)$ be the resulting sequence．Finally，define $\operatorname{MVPD}(w)$ as the set of MVPDs with column－to－row code $\alpha^{\prime}(w)$ ．

Example 3．4．Take $w \in S_{n}$ such that $w^{-1}$ has one－line notation 3142．We have $\alpha^{\prime}(w)=$ $(0,1,0,2)$ ．The set $\operatorname{MVPD}(w)$ has the following three elements：


We describe a bijection from $\operatorname{PD}(w)$ to $\operatorname{MVPD}(w)$ ．Take $P \in \operatorname{PD}(w)$ for some $w \in S_{n}$ ．Let $p_{1}, \cdots, p_{k}$ be the left－to－right of $w^{-1}$ ．We may remove the pipes $p_{1}, \cdots, p_{k}$ ．If a $\boxplus$ becomes a $\square$ after the removal，we mark it as $\square$ ．Let $\Phi(P)$ be the resulting tiling．

Example 3．5．Suppose $n=8$ ．Consider $w \in S_{8}$ where $w^{-1}$ has one－line notation 12547386. The left－to－right maximums of $w^{-1}$ are $1,2,5,7$ and 8 ．We consider the following $P \in \operatorname{PD}(w)$ where pipe 1 ，pipe 2 ，pipe 5 ，pipe 7 and pipe 8 are colored red．


Readers may check $\Phi(P)$ would be the MVPD in Example 3．2．
Proposition 3．6．The map $\Phi$ is a bijection from $\operatorname{PD}(w)$ to $\operatorname{MVPD}(w)$ that preserves wty $(\cdot)$
Proof．Take any $P \in \operatorname{PD}(w)$ and consider $\Phi(P)$ ．First，if both $P$ and $\Phi(P)$ has pipe $p$ ，then it travels the same in $P$ and $\Phi(P)$ ．We now check $\Phi(P) \in \operatorname{MVPD}(w)$ ．
－We make sure $\Phi(P)$ has no $\square$ ．Suppose to the contrary that $\Phi(P)$ at $(i, j)$ is a $\square$ ， then $P$ must have a $\boxplus$ at $(i, j)$ ．Let pipe $p$（resp．$q$ ）be the pipe going horizontally （resp．vertically）in this tile．Then we know pipe $p$ is removed by $\Phi$ ，so $p$ is a left－to－ right maximum of $w^{-1}$ ．However，since pipe $p$ and pipe $q$ crossed in this tile，we know $p<q$ and $q$ appears on the left of $p$ in the one－line notation of $w^{-1}$ ，contradicting to $p$ being a left－to－right maximum of $w^{-1}$ ．
－Assume pipe $p$ has $\square$ at $(i, j)$ of $\Phi(P)$ ，we check pipe $p$ has a $\square$ before．We know $P$ has a $⿴ 囗 十$ at $(i, j)$ ．Let pipe $q$ be the other pipe in $(i, j)$ of $P$ ，so this pipe is removed by $\Phi$ ．We know pipe $p$ and pipe $q$ already crossed before $(i, j)$ in $P$ ，say at $\left(i^{\prime}, j^{\prime}\right)$ ． Then after removing pipe $q$ ，the $\left(i^{\prime}, j^{\prime}\right)$ becomes $\boxminus$ in $\Phi(P)$ ．

We have checked $\Phi(P)$ is a valid MVPD．Clearly，$\Phi(P)$ has column－to－row code $\alpha^{\prime}(w)$ ，so it is in $\operatorname{MVPD}(w)$ ．We check $\Phi$ preserves wty $(\cdot)$ ．Take a $\boxplus$ in $P$ ．We check it becomes a weighty tile in $\Phi(P)$ ．Say pipe $p$ exits from the top and pipe $q$ exits from the right of this田．By 3．3，it is impossible that both pipe $p$ and pipe $q$ are removed by $\Phi$ ，so this $⿴ 囗 十$ will not become a $\boxtimes$ ．It also cannot become a $\square$ ：If so，we know $q$ is a left－to－right maximum in $w^{-1}, q<p$ ，and $q$ ends up on the right of $p$ in $w^{-1}$ ．This is a contradiction．It is also obvious that this $⿴ 囗 十$ cannot be mapped to $\square$ or $\boxtimes$ by the rules of $\Phi$ ．Therefore $\Phi$ maps $\boxplus$ to weighty tiles．On the other hand，for any $\boxtimes$ in $P$ ，they cannot be mapped to $\boxminus$ ，$\boxplus$ ，or $\square$ by the rules of $\Phi$ ．

Thus，$\Phi$ is a wty $(\cdot)$ preserving map from $\operatorname{PD}(w)$ to $\operatorname{MVPD}(w)$ ．It remains to construct its inverse．Take $M \in \operatorname{MVPD}(w)$ ．We change the cell $(i, j)$ based on the following：
－If it is $\boldsymbol{\square}$ or $\boxminus$ ，it becomes $\boxplus$ ．
－If it is $\rrbracket$ ，we know $i+j \leqslant n+1$ ．If $i+j<n$ ，we change it into $\boxtimes$ ．
－If it is $\square$ ，we turn it into $\square$ ．
－Finally，suppose it is $\square$ ．If $i+j<n$ ，we turn it into $\boxtimes$ ．If $i+j=n$ ，we turn it into $\square$.

Clearly，we obtain a PD $P$ by adding pipes to each tile．We claim for each pipe in $M$ ，it goes in the same way in both $P$ and $M$ ．In addition，the added pipes in $P$ belong to pipes which do not exist in $M$ ．We prove by induction on the tiles from bottom to top，and left to right in each row．Consider the tile $(i, j)$
－If $(i, j)$ is a $\boxminus$ containing pipe $p$ in $M$ ，it becomes a $\boxplus$ in $P$ ．We need to verify that pipe $p$ goes horizontally in $(i, j)$ of $P$ ．Let pipe $q$ be the pipe entering from the bottom of $(i, j)$ in $P$ ，so pipe $q$ does not exist in $P$ ．Assume toward contradiction that $p$ does not go horizontally in $(i, j)$ ．Then pipe $p$ and pipe $q$ have already crossed， where the pipe $p$ travels vertically．Then the corresponding cell in $M$ would be a $\boldsymbol{\square}$ ， which is impossible．
－If $(i, j)$ is a containing pipe $p$ in $M$ ，it becomes a $\boxplus$ in $P$ ．We need to verify that pipe $p$ does not go vertically in $(i, j)$ of $P$ ．Let pipe $q$ be the pipe entering from the left of $(i, j)$ in $P$ ，so pipe $q$ does not exist in $P$ ．We need to show pipe $p$ and $q$ have crossed before．Since $(i, j)$ is $\square$ in $M$ ，we may find a $\boxminus$ containing pipe $p$ under row $i$ ．In $P$ ，it becomes a $⿴ 囗 十$ where pipe $p$ crosses with some added pipe，say pipe $t$ ．If $t=q$ ，we are done．Otherwise，we know the added pipes cannot cross．Thus，the three pipes enter row $i$ with the order $t, q, p$ from left to right．By Lemma 2．4，we know pipe $q$ and $p$ have crossed．
－The other cases of $(i, j)$ is straightforward to check．

Say $P \in P D(u)$ ．The claim above says the $k^{\text {th }}$ entry of $\alpha^{\prime}(w)$ ，if non－zero，agrees with $u^{-1}(k)$ ．Since the added pipes are not crossing，we know $u^{-1}(k)$ is obtained from $\alpha^{\prime}(w)$ by turning 0 s into the missing numbers in increasing order，which yields $w^{-1}$ ．Thus，$u=w$ and the map defined above sends $\operatorname{MVPD}(w)$ to $\operatorname{PD}(w)$ ．It is clearly the inverse of $\Phi$ ．

Corollary 3.7. For $w \in S_{n}$, we have

$$
\begin{aligned}
\mathfrak{G}_{w}(\boldsymbol{x}) & =\sum_{M \in \operatorname{MVPD}(w)}(-1)^{|\operatorname{wty}(M)|-\ell(w)} \operatorname{wt}_{M}(\boldsymbol{x}), \\
\mathfrak{G}_{w}(\boldsymbol{x}, \boldsymbol{y}) & =\sum_{M \in \operatorname{MVPD}(w)}(-1)^{|\operatorname{wty}(M)|-\ell(w)} \operatorname{wt}_{M}(\boldsymbol{x}, \boldsymbol{y}) .
\end{aligned}
$$

Proof. Follows from (1) and Proposition 3.6.

## 4. Bumpless Vertical-less Pipedreams

4.1. Describing the BVPD formula. We introduce bumpless vertical-less pipedreams (BVPD). They consist of tilings where the following five types of tiles are placed

$$
\square, \quad \boxminus, \quad \boxplus, \quad \square, \quad \square
$$

on an $n \times(n-1)$ grid. The pipes of a BVPD enter from the left edge and exit from the top edge. We trace pipes from left to top. For each $\boxplus$, we trace the pipes in the same way as PDs and MVPDs. We name the pipe entering from row $p$ as pipe $p$. When drawing a BVPD, we omit blank rows on the bottom and blank columns to the right.

Example 4.1. Let $n=6$. The following is a BVPD

with pipe 2 going to column 5 and pipe 3 going to column 4 .
The column-to-row code of a BVPD is a sequence of $n-1$ numbers, defined similarly as that of a MVPD. The column-to-row code of the BVPD in Example 4.1 is ( $0,0,0,3,2$ ).

Take $w \in S_{n}$ be inverse fireworks. We obtain a sequence $\alpha(w)$ as follows. We start from the sequence $\left(w^{-1}(1), \cdots, w^{-1}(n)\right)$ and set the first number in each decreasing run to be 0 . Then $\alpha(w)$ is obtained by removing the first entry. Notice that $\alpha(w)$ can be obtained from $\alpha^{\prime}(w)$ by removing the first entry. Let $\operatorname{BVPD}(w)$ be the set of all BVPDs with column-to-row code $\alpha(w)$.

Example 4.2. Say $n=6$ and $w$ has one-line notation 165234. Thus, $w^{-1}$ has one-line notation 145632 where $1,4,5$ and 6 are the first numbers in the decreasing runs. We have $\alpha(w)=(0,0,0,3,2)$, so $\operatorname{BVPD}(w)$ consists of all BVPDs with pipe 2 going to column 5 , pipe 3 going to column 4 , and there are no other pipes. There are six such BVPDs:



Finally，define the weighty tiles of a BVPD $B$ ，denoted as wty $(B)$ ，as the set of $(i, j)$ that is $\boxplus, \rrbracket$ or $\boxminus$ in $B$ ．Let the weight of $B$ ，denoted as $\mathrm{wt}_{B}(\mathbf{x})$ ，be the monomial $\Pi_{(i, j) \in \mathrm{wty}(B)} x_{i}$ ． We write the weight of each BVPD under itself in Example 4．2．

Theorem 4．3．For inverse fireworks $w$ ，we have

$$
\widehat{\mathfrak{G}}_{w}(\boldsymbol{x})=\sum_{B \in \operatorname{BVPD}(w)} \mathrm{wt}(B) .
$$

Continuing on Example 4．2．If $w$ has one－line notation 165234，then

$$
\widehat{\mathfrak{G}}_{w}(\mathbf{x})=x_{1}^{2} x_{2}^{4} x_{3}^{3}+x_{1}^{3} x_{2}^{3} x_{3}^{3}+x_{1}^{4} x_{2}^{2} x_{3}^{3}+x_{1}^{3} x_{2}^{4} x_{3}^{2}+x_{1}^{4} x_{2}^{3} x_{3}^{2}+x_{1}^{4} x_{2}^{4} x_{3}^{1}
$$

Theorem 4．4．For $w$ inverse fireworks，there exists a bijection $\Psi$ from $\operatorname{BVPD}(w)$ to $\widehat{\operatorname{PD}(w)}$ that preserves the positions of weighty tiles．

Roughly speaking，for $B \in \operatorname{BVPD}(w), \Psi(B)$ is the pipedream with a $⿴ 囗 十$ at row $i$ column $j$ for each $(i, j) \in \mathrm{wty}(B)$ and no $\boxplus$ elsewhere．This result also characterizes the pipedreams of $w$ with the maximal number of $⿴ 囗 十$ when $w$ is inverse fireworks．

4．2．Proofs of Theorem 4.3 and Theorem 4．4．We start with one simple property on the number of weighty tiles in a MVPD．For $w \in S_{n}$ ，define $r(w):=\sum_{i} i-1$ where $i$ ranges over all number such that the $i^{\text {th }}$ number in $\alpha^{\prime}(w)$ is non－zero．

Lemma 4．5．Take $M \in \operatorname{MVPD}(w)$ ．Let $k$ be the number of $\boxtimes$ and $\square$ in $M$ ．We have $|w \operatorname{ty}(M)|=r(w)-k$ ．
Proof．We first associate each tile in $M$ that is not $\square$ or $\square$ to each pipe．These tiles must be $\boxminus, \boxplus, \square, \square$ or $\boxtimes$ ．We associate each such tile to the pipe that exits from the right．

Take an arbitrary pipe $p$ and suppose it goes to column $c_{p}$ ．In other words，the $c^{\text {th }}$ number in $\alpha^{\prime}(w)$ is $p$ ．For each column $i$ ，we count the number of cells associated with pipe $p$ in this column：
－If the pipe $p$ exits column $i$ and goes to column $i+1$（i．e． $1 \leqslant i<c_{p}$ ），there is exactly one tile associated with pipe $p$ in column $i$ ．
－Otherwise（i．e．$i \geqslant c_{p}$ ），there is no tile associated with pipe $p$ ．
Now there are $c_{p}-1$ tiles associated with the pipe $p$ ．Let $k_{p}$ be the number of $\square$ and $⿴ 囗 十$ associated with pipe $p$ ．The number of weighty tiles associated with $p$ is $\left(c_{p}-1\right)-k_{p}$ ．We have

$$
\begin{aligned}
|\mathbf{w t y}(M)| & =\sum_{\operatorname{pipes} p \text { in } M} \mathrm{wt}(p)=\sum_{\operatorname{pipes} p \text { in } M}\left(c_{p}-1\right)-k_{p} \\
& =\sum_{\operatorname{pipes} p \text { in } M}\left(c_{p}-1\right)-\sum_{\operatorname{pipes} p \text { in } M} k_{p}=r(w)-k .
\end{aligned}
$$

Let $\widehat{\operatorname{MVPD}}(w)$ be the subset of $\operatorname{MVPD}(w)$ with maximal number of weighty tiles．Recall that $\operatorname{raj}(w)$ is the degree of $\mathfrak{G}_{w}(\mathbf{x})$ ，so an element of $\operatorname{MVPD}(w)$ has $\operatorname{raj}(w)$ weighty tiles．We can describe $\widehat{\operatorname{MVD}( } w)$ of inverse fireworks $w$ as follows．
 ments in $\operatorname{MVPD}(w)$ without $\square$ and $\Xi$ ．

Proof．By Lemma 4．5，it remains to show $r(w)$ is the maximal number of weighty tiles of an element in $\operatorname{MVPD}(w)$ ，which is $\operatorname{raj}(w)$ ．By Corollary 2．6， $\operatorname{raj}(w)=\operatorname{raj}\left(w^{-1}\right)$ ．Since $w^{-1}$ is fireworks，by Proposition 2．5， $\operatorname{raj}\left(w^{-1}\right)=\operatorname{maj}\left(w^{-1}\right)$ ．It remains to check $r(w)=\operatorname{maj}\left(w^{-1}\right)$ ．

Recall that $r(w)=\sum_{i \in I}(i-1)$ ，where $I=\left\{i: i^{\text {th }}\right.$ entry of $\alpha^{\prime}(w)$ is not 0$\}$ ．In other words， $I$ consists of all $i$ such that $w^{-1}(i)$ is not a left－to－right maximum of $w^{-1}$ ．Since $w^{-1}$ is fireworks，$I$ consists of $i$ such that $w^{-1}(i)$ is not the first number in its decreasing run．Then we have

$$
\begin{aligned}
\{i-1: i \in I\} & =\left\{j: w^{-1}(i) \text { is not the last number in its decreasing run }\right\} \\
& =\left\{j: w^{-1}(j)>w^{-1}(j+1)\right\} .
\end{aligned}
$$

Thus，$r(w)=\sum_{i \in I}(i-1)=\sum_{j: w^{-1}(j)>w^{-1}(j+1)} j=\operatorname{maj}\left(w^{-1}\right)$ ．
Corollary 4．7．The first column of any $M \in \mathrm{M} \widehat{\mathrm{VPD}( } w)$ only consists of $\square$ and $\boxminus$ ．
Proof．Suppose not．Say pipe $p$ has a $\square$ in column 1 of $M$ ．We know $w^{-1}(1)$ is a left－to－right maximum in $w^{-1}$ ，so the first entry in $\alpha^{\prime}(w)$ is 0 ．In other words，pipe $p$ must exit column 1．Find the cell in column 1 where pipe $p$ exits enters from the bottom and exits from the right．By $M \in \widehat{\operatorname{MPD}( } w)$ and Lemma 4．6，this cell can only be $\boxplus$ where the two pipe or $\boldsymbol{\square}$ ． It cannot be a $⿴ 囗 十$ since pipe $p$ has not crossed with the pipe entering from the left．It cannot be a $\square$ since pipe $p$ does not have a $\boxminus$ before．Contradiction．

Now it remains to establish a bijection from $\operatorname{MVPD}(w)$ to $\operatorname{BVPD}(w)$ ．We describe the map $\Phi^{M \rightarrow B}$ as follows．Take $M \in \operatorname{MVPD}(w)$ ，we remove its first column and change all into $\square$ ，obtaining a tiling $B$ ．The inverse of this map，denoted as $\Phi^{B \rightarrow M}$ is also straightforward： Add a column on the left of $B$ consisting of $\boxminus$ and $\square$ and change all $\square$ in $B$ into $\square$ ．

Proposition 4．8．The maps $\Phi^{M \rightarrow B}$ and $\Phi^{B \rightarrow M}$ are bijections between $\widehat{\operatorname{MPD}(w)}$ and $\operatorname{BVPD}(w)$ that preserve wty（•）

Proof．Say $\Phi^{M \rightarrow B}$ sends $\left.M \in \widehat{\operatorname{MPD}( } w\right)$ to $B$ ．Since $M$ has neither $\square$ nor $\boxtimes, B$ is a BVPD． Recall that $\alpha(w)$ is obtained from $\alpha^{\prime}(w)$ by removing the first 0 ．Since $M$ has column－to－row code $\alpha^{\prime}(w)$ ，we know $B$ has column－to－row code $\alpha(w)$ ，so $B \in \operatorname{BVPD}(w)$ ．The two maps are clearly inverses of each other．To show the bijections preserve wty $(\cdot)$ ，we present the following example．

Example 4．9．The left diagram is $M \in \widehat{\operatorname{MPD}( } w)$ and the left diagram is $\Phi^{M \rightarrow B}(M)=B \in$ $\operatorname{BVPD}(w)$ ．Their weighty tiles（highlighted yellow）agree．


Now we prove the main results of this section．
Proof of Theorem 4．3．By Corollary 3．7，$\widehat{\mathfrak{G}}_{w}(\mathbf{x})=\sum_{M \in \mathrm{MVPD}(w)} \mathrm{wt}_{M}(\mathbf{x})$ ．Then by Proposi－ tion 4．8，$\sum_{M \in \operatorname{MVPD}(w)} \mathrm{wt}_{M}(\mathbf{x})=\sum_{B \in \operatorname{BVPD}(w)} \mathrm{wt}_{B}(\mathbf{x})$ ．

Proof of Theorem 4．4．Take $B \in \operatorname{BVPD}(w)$ ．To obtain $P \in \widehat{\operatorname{PD}(w)}$ ，we simply apply $\Phi^{B \rightarrow M}$ to $B$ ，followed by the bijection from $\operatorname{MVPD}(w)$ to $\operatorname{PD}(w)$ ．Both maps preserve wty $(\cdot)$ ，so $\mathrm{wty}(B)=\mathrm{wty}(P)$ ．

## 5．Proof of Conjecture 1．1 for inverse fireworks permutations

In this section，we prove Conjecture 1.1 for inverse fireworks $w$ ．Our approach is construc－ tive：For $M \in \operatorname{MVPD}(w) \backslash \operatorname{MVPD}(w)$ ，we construct $M^{\prime}$ such that $\mathrm{wt}_{M}(\mathbf{x}) x_{i}=\mathrm{wt}_{M^{\prime}}(\mathbf{x})$ for some $i$ using＂droop moves＂．In Section 5．1，we develop some general properties of MVPDs and define droop moves on MVPDs．Then we give the construction in Section 5．2．

5．1．Properties of MVPD．Let $w \in S_{n}$ be an arbitrary permutation in this section．We start with two observations on MVPD $(w)$ ．

Lemma 5．1．Take $M \in \operatorname{MVPD}(w)$ ．For every pipe，we can find $a \boxminus$ in $M$ containing that pipe．

Proof．Consider the pipe from row $r$ of $M$ ．We know $r$ appears in $\alpha^{\prime}(w)$ ，so $r$ is not a left－ to－right maximum in $w^{-1}$ ．Say $m>r$ is a number on the left of $r$ in $w^{-1}$ ．In the pipedream corresponding to $M$ ，there must be a $\boxplus$ where the pipe from row $r$ goes from left to right and the pipe from row $m$ goes from bottom to top．To obtain $M$ from this pipedream，we remove the pipe from row $m$ ，so this tile becomes a $\boxminus$ ．

We say a $⿴ 囗 十$ of $M$ is a real crossing if its two pipes really cross in it（i．e．the pipe entering from the bottom exits from top）．Otherwise，we say the $\boxplus$ is a fake crossing．

Lemma 5．2．Take $M \in \operatorname{MVPD}(w)$ ．Say pipe $p$ and pipe $q$ have a real crossing in $(i, j)$ and a fake crossing in $\left(i^{\prime}, j^{\prime}\right)$ ．We consider the region enclosed by the two pipes from $(i, j)$ to $\left(i^{\prime}, j^{\prime}\right)$ ．For any pipe that appears in this region，it must cross with both pipe $p$ and pipe $q$ ．

Proof．For a pipe $t$ to enter or exit this region，it must cross with pipe $p$ or pipe $q$ ．Since two pipes cannot cross more than once，pipe $t$ must cross both pipe $p$ and pipe $q$ ．

Next，we define the droop moves on MVPDs，which look similar to the droop moves on bumpless pipedreams introduced in［LLS21］．
Definition 5．3．Take $M \in \operatorname{MVPD}(w)$ ．We define $\operatorname{droop}_{(i, j)}(M)$ if the following are all satisfied
－The tile $(i, j)$ contains a pipe entering from the bottom and exits from the right（i．e． It is a $\square, \square, \square$ or a fake crossing）．
－The tile $(i, j+1)$ is a $\boxminus$ ．
－Let $i^{\prime}>i$ be the smallest such that $\left(i^{\prime}, j\right)$ is not $\boxplus$ ．Then $\left(i^{\prime}, j\right)$ is a $\rrbracket$ ．
For each $i<r<i^{\prime}$ ，we know $(r, j)$ is a $⿴ 囗 十$ ．A simple induction would imply that $(r, j+1)$ has no pipe entering from the bottom．Thus，$(r, j+1)$ is $\boxminus$ and $\left(i^{\prime}, j+1\right)$ is $\square, \square$ or $\square$ ． The operation $\operatorname{droop}(\cdot)$ does the following to column $j$ and $j+1$ between row $i$ and row $i^{\prime}$ ．
－Change $(i, j)$ from $\boxtimes$ or fake crossing to $\boxtimes$ ．Change $(i, j)$ from $\square$ or to $\square$ ．
－Change $(i, j+1)$ from $\boxminus$ to $\square$ ．
－For $i<r<i^{\prime}$ ，change $(r, j)$ from $\boxplus$ to $\boxminus$ and change $(r, j+1)$ from $\boxminus$ to $\boxplus$ ．
－Change $\left(i^{\prime}, j\right)$ from $\square$ to $\boxminus$ ．
－Change $\left(i^{\prime}, j+1\right)$ from $\square$ or to $\square$ ．Change $\left(i^{\prime}, j+1\right)$ from $\square$ to $\rrbracket$ ．
We also define $\operatorname{droop}_{(i, j)}^{\prime}(M)$ on such $(i, j)$ and $M$ ．It first performs droop ${ }_{(i, j)}$ ．Then notice that the pipe in $(i, j+1)$ of $\operatorname{droop}_{(i, j)}(M)$ must have a $\boxminus$ in $(r, j)$ for some $i<r \leqslant i^{\prime}$ ．We may mark the pipe in $(i, j+1)$ ，obtaining a valid MVPD droop ${ }_{(i, j)}^{\prime}(M)$ ．

Example 5．4．We give two examples of the effect of $\operatorname{droop}_{(i, j)}^{\prime}$ and $\operatorname{droop}_{(i, j)}$ ．


Lemma 5．5．Take $M \in \operatorname{MVPD}(w)$ ．Then $\operatorname{droop}_{(i, j)}(M)$ and $\operatorname{droop}_{(i, j)}^{\prime}(M)$ are both in $\operatorname{MVPD}(w)$ if they are defined．

Proof．Let $i^{\prime}>i$ be the smallest such that $\left(i^{\prime}, j\right)$ is not $⿴ 囗 十$ ．We just need to show that the same pipe exits from the right edge of $(r, j+1)$ for $i \leqslant r<i^{\prime}$ in $M$ and $\operatorname{droop}_{(i, j)}(M)$ ．To prove this，we claim：For $i<r \leqslant i^{\prime}$ ，if a pipe exits from the top of $(r, j)$ in $M$ ，then the same pipe exits from the top of $(r, j+1)$ in $\operatorname{droop}_{(i, j)}(M)$ ．We prove by induction on $r$ ．The base case when $r=i^{\prime}$ is immediate．Now suppose $i<r<i^{\prime}$ ．Say pipe $p$ enters $(r, j)$ from the left and pipe $q$ enters $(r, j)$ from the bottom in $M$ ．Then pipe $\max (p, q)$ exits from the the top of $(r, j)$ ．The other pipe exits from the right of $(r, j+1)$ ．By our inductive hypothesis，pipe $p$ enters $(r, j+1)$ from the left and pipe $q$ enters $(r, j+1)$ from the bottom in $\operatorname{droop}_{(i, j)}(M)$ ． Pipe $\max (p, q)$ exits from the the top of $(r, j+1)$ ，and the other pipe exits from the right． Our inductive step is finished．

Finally，we study a special family of MVPDs．
Definition 5．6．A $M \in \operatorname{MVPD}(w)$ is called saturated if it satisfies both of the following．
－For any $\square$ in $M$ ，the pipe in it does not have $\boxminus$ before．
－For any $\boxtimes$ ，the two pipes in it do not cross in $M$ ．

In other words，an $M \in \operatorname{MVPD}(w)$ is not saturated if we can turn one of its $⿴$ to $⿴ 囗 十 \square$ to and still remain in $\operatorname{MVPD}(w)$ ．

Lemma 5．7．Take a saturated $M \in \operatorname{MVPD}(w)$ ．Say a pipe $p$ enters the tile $(i, j)$ from the bottom and exits from the right．Then $(i, j+1)$ cannot be a real crossing．

Proof．Suppose there exists such $(i, j)$ ．We pick one such $(i, j)$ where $i$ is maximal．Say pipe $p$ enters from the bottom of $(i, j)$ and say it crosses with pipe $q$ in $(i+1, j)$ ．We know these two pipes have not crossed before $(i, j+1)$ ．Moreover，since $M$ is saturated，there is no $\boxtimes$ in $M$ involving pipe $p$ and pipe $q$ ．As a conclusion，under row $i$ ，there is no tile containing both pipe $p$ and pipe $q$ ．

Find $i^{\prime}>i$ such that pipe $p$ enters on the left edge of $\left(i^{\prime}, j\right)$ ．We know pipe $p$ goes from bottom to top of $(r, j)$ for $i<r<i^{\prime}$ ．Say pipe $q$ enters on the left edge of $\left(i^{\prime \prime}, j+1\right)$ ．We have $i^{\prime \prime}>i$ since otherwise，$\left(i^{\prime \prime}, j\right)$ would be a tile containing both pipe $p$ and pipe $q$ ．Pipe $q$ goes from bottom to top of $\left(i^{\prime \prime}, j+1\right)$ ，so this tile is a real crossing．Consider the tile $\left(i^{\prime}, j\right)$ ． It contains pipe $p$ which enters on the left and exits on the top．Thus，it also must contains a pipe entering from the bottom and exits on the right．We reach a contradiction since we picked the maximal $i$ ．

Here is an illustration of the proof of Lemma 5．7．We make pipe $p$ green and pipe $q$ red．


5．2．Construction．Fix inverse fireworks $w$ in throughout this section．For each $M \in$ $\operatorname{MVPD}(w) \backslash \operatorname{MVPD}(w)$ ，our goal is to construct $M^{\prime}$ such that $\mathrm{wt}_{M^{\prime}}(\mathbf{x})=\mathrm{wt}_{M}(\mathbf{x}) x_{i}$ for some $i$ ． If $M$ is not saturated，we can find the $M^{\prime}$ easily：Say $M$ has an $\square$ and the pipe in it has a $\boxminus$ before，we simply mark the $\square$ and obtain $M^{\prime}$ ．Otherwise，say $M$ has a $\boxtimes$ where the two pipes in it cross somewhere else．We may turn this $\boxtimes$ into $\boxplus$ and the resulting MVPD is still in $\operatorname{MVPD}(w)$ ．It remains to consider saturated $M$ ．Our construction relies on the operator $\operatorname{droop}_{i, j}^{\prime}(\cdot)$ ，which requires us to find an occurrence of $\square \square$ or $\square$ in $M$ ．That is，a $\square$ or $\square$ with a $\boxminus$ immediately on its right．

Lemma 5．8．Take a saturated $M \in \operatorname{MVPD}(w) \backslash \operatorname{MVPD}(w)$ ．In $M$ ，there exists $\square \square \square$ or
Proof．Since $M \notin \operatorname{MVPD}(w)$ ，by Lemma 4．6，$M$ must have a $\boxtimes$ or $\square$ ．Let $(i, j)$ be the highest，or one of the highest，such tile．We prove $(i, j+1)$ must be $\boxminus$ by contradiction． Suppose $(i, j+1)$ is not a $\boxminus$ ．Let pipe $p$ be the pipe that enters $(i, j)$ from the bottom and exits on the right．By Lemma 5．7，$(i, j+1)$ cannot be a real crossing．Thus，$(i, j+1)$ can be a fake crossing，a $\boxtimes$ ，or a $\rrbracket$ ．In any case，pipe $p$ must exits on the top of $(i, j+1)$ ．Then we present two different arguments based on whether $(i, j)$ is $⿴ 囗 \square$ or ．Both arguments follow the following three steps：
－Step 1：Show pipe $p$ must exits column $j+1$ ．Say it exits from the right edge of $\left(i^{\prime}, j+1\right)$ for some $i^{\prime}<i$ ．

- Step 2: We know $\left(i^{\prime}, j+1\right)$ cannot be $\square$ or by how we picked $(i, j)$. We show $\left(i^{\prime}, j+1\right)$ cannot be an $\boldsymbol{\square}$, so it must be a fake crossing.
- Step 3: Find a contradiction.

We start with the case where $(i, j)$ is a bump. Let pipe $q$ be the pipe exiting from the top of $(i, j)$. Since $M$ is saturated, we know pipe $p$ and pipe $q$ never cross in $M$, so $q<p$. Now we perform the three steps and eventually show pipe $p$ and pipe $q$ must cross, which would be a contradiction.

- Step 1: Suppose pipe $p$ does not exit column $j+1$. Since pipe $p$ and $q$ cannot cross, pipe $q$ does not exit column $j$. In other words, $\alpha(j+1)=p$ and $\alpha(j)=q$. Then $w^{-1}(j+2)=p$ and $w^{-1}(j+1)=q$. Since $q<p, p$ is actually the first number in its decreasing run in $w^{-1}$, so $p$ cannot appear in $\alpha(w)$. We reach a contradiction, so pipe $p$ must exit column $j+1$.
- Step 2: We know pipe $p$ goes from the bottom to top in $(r, j+1)$ for $i^{\prime}<r<i$. Since pipe $q$ cannot cross with pipe $p$, it must also go from bottom to top in $(r, j)$ for $i^{\prime}<r<i$. Thus, pipe $q$ enters $\left(i^{\prime}, j\right)$ from the bottom. The tile $\left(i^{\prime}, j\right)$ must have some pipe exiting from the right. Thus, $\left(i^{\prime}, j+1\right)$ has a pipe entering from the left, so it cannot be $\boldsymbol{6}$. It must be a fake crossing.
- Step 3: Let pipe $t$ be the pipe that enters $\left(i^{\prime}, j+1\right)$ from the left. Since $\left(i^{\prime}, j+1\right)$ is a fake crossing, pipe $t$ and pipe $p$ must have a real crossing under row $i$. Then pipe $t$ must exits row $i$ on the left of pipe $p$. Since pipe $p$ exits row $i$ on column $j+1$ and pipe $q$ exits row $i$ on column $j$, we know pipe $t$ exits row $i$ on the left of column $j$. Now consider the region enclosed by pipe $p$ and $t$ from their real crossing to $\left(i^{\prime}, j+1\right)$. Pipe $q$ appears in this region. By Lemma 5.2, pipe $q$ crosses with pipe $p$. Contradiction.
Now assume $(i, j)$ is $\square$. By $M$ is saturated, we know pipe $p$ does not have a $\boxminus$ before $(i, j)$. We perform the three steps.
- Step 1: If pipe $p$ does not exits column $j+1$, then it does not have a $\boxminus$ in $M$, contradicting Lemma 5.1.
- Step 2: In $\left(i^{\prime}, j\right)$, the pipe $p$ still does not have a $\boxminus$ yet, so $\left(i^{\prime}, j\right)$ cannot be $\boldsymbol{\square}$. It must be a fake crossing.
- Step 3: Say $\left(i^{\prime}, j+1\right)$ is a fake crossing between pipe $p$ and pipe $t$. Pipe $p$ must have a real crossing under row $i$ where pipe $p$ goes horizontally. In other words, we can find a real crossing $\left(r_{+}, c_{+}\right)$where pipe $p$ goes horizontally with $r_{+}>i$. Take the $\left(r_{+}, c_{+}\right)$ where $c_{+}$is maximal. Thus, from $\left(r_{+}, c_{+}\right)$to $(i, j+1)$, the pipe $p$ is not allowed to travel horizontally in any tile. In other words, if pipe $p$ enters a tile from the left, it must exit from the top.

Next, we argue for $i \leqslant r \leqslant r_{+}$, when pipe $p$ exits row $r$, there is a pipe exiting from the cell on its left, which has already crossed with pipe $p$.

We prove our claim by induction. The base case is when $r=r_{+}$. We know $\left(r_{+}, c_{+}\right)$ is a real crossing. Since pipe $p$ enters $\left(r_{+}, c_{+}+1\right)$ from the left, it exits row $r_{+}$from $\left(r_{+}, c_{+}+1\right)$. Indeed, $\left(r_{+}, c_{+}\right)$has a pipe exiting from the top, which just crossed with pipe $p$. Now take $i \leqslant r<r_{+}$. Say pipe $p$ enters from the bottom of $(r, c)$. By our inductive hypothesis, another pipe enters $(r, c-1)$ from the bottom. Say it is pipe $s$ If pipe $p$ goes vertically in $(r, c)$, pipe $s$ must go vertically in $(r, c-1)$ since if it exits on the right, $(r, c)$ would be a fake crossing. Now suppose pipe $p$ exits $(r, c)$ on
the right. Since $(r, c-1)$ has a pipe entering from the bottom, it must has a pipe exiting from the right. Then $(r, c)$ can be a fake crossing or a $Q$. Consider the region enclosed by pipe $t$ and pipe $p$ from their real crossing to $\left(i^{\prime}, j\right)$. The other pipe in $(r, c)$ is either pipe $t$, or lies in this region. In either case, it must cross with pipe $p$. Since $M$ is saturated, $(r, c)$ is not a $\boxtimes$, so it is a fake crossing. Pipe $p$ exits from the top of $(r, c+1)$ and some pipe that has crossed with it exits from the top of $(r, c)$.

Finally, our claim implies when pipe $p$ exits $(i, j+1)$ from the top, there must be a pipe that exits $(i, j)$ from the top. This contradicts to our assumption that $(i, j)$ is $\square$.

Now we describe our algorithm. Take a saturated $M \in \operatorname{MVPD}(w) \backslash \widehat{\operatorname{MPD}(w)}$. By Lemma 5.8, we know $M$ must have a $\square$ or $\square$. We let $(i, j)$ and $(i, j+1)$ be the lowest such occurrence where we first maximize $i$, and then $j$. We check $\operatorname{droop}_{(i, j)}^{\prime}(M)$ is defined. The first two conditions in Definition 5.3 are immediate. For the last condition, we let $i^{\prime}>i$ be the smallest such that $\left(i^{\prime}, j\right)$ is not $\boxplus$. It can be $\boxtimes$ or $\rrbracket$. Assume it is a $\boxtimes$ toward contradiction. Since $\left(i^{\prime}-1, j+1\right)$ is $\boxminus$, we know $\left(i^{\prime}, j+1\right)$ is also a $\boxminus$. This contradicts the maximality of $i$. Thus, $\left(i^{\prime}, j\right)$ is a $\square$ and $\operatorname{droop}_{(i, j)}(M)$ is well-defined.

Next, the algorithm computes $\operatorname{droop}_{(i, j)}^{\prime}(M)$, which is in $\operatorname{MVPD}(w)$ by Lemma 5.5. We compare the weighty tiles of $M$ and $\operatorname{droop}_{(i, j)}^{\prime}(M)$ :

- The tile $(i, j)$ is not weighty in $M$ and $\operatorname{droop}_{i, j}^{\prime}(M)$. The tile $(i, j+1)$ is weighty in $M$ and $\operatorname{droop}_{i, j}^{\prime}(M)$.
- For $i<r<i^{\prime}$, the tile $(r, j)$ and $(r, j+1)$ are weighty in $M$ and $\operatorname{droop}_{i, j}^{\prime}(M)$.
- The tile $\left(i^{\prime}, j\right)$ is not weighty in $M$ but becomes weighty in $\operatorname{droop}_{i, j}^{\prime}(M)$. The tile $\left(i^{\prime}, j+1\right)$ could be either weighty or not in $M$, but is not weighty in $\operatorname{droop}_{i, j}^{\prime}(M)$.
If $\operatorname{droop}_{i, j}^{\prime}(M)$ has one more weighty tile than $M$, we let $M^{\prime}=\operatorname{droop}_{i, j}^{\prime}(M)$ and terminate. Then $\mathrm{wt}_{M^{\prime}}(\mathbf{x})=\mathrm{wt}_{M}(\mathbf{x}) x_{i^{\prime}}$. Otherwise,

$$
\mathrm{wty}\left(\operatorname{droop}_{i, j}^{\prime}(M)\right)=\left(\mathrm{wty}(M) \backslash\left\{\left(i^{\prime}, j+1\right\}\right) \cup\left\{\left(i^{\prime}, j\right)\right\},\right.
$$

so $\operatorname{droop}_{i, j}^{\prime}(M)$ and $M$ have the same number of weighty tiles. If droop ${ }_{i, j}^{\prime}(M)$ is not saturated, then we change a $\square$ or a $\boxtimes$ into a weighty tile and obtain $M^{\prime}$. Otherwise, we update the variable $M$ into $\operatorname{droop}_{(i, j)}^{\prime}(M)$ and repeat the algorithm.

It remains to show the algorithm eventually terminates. Let $M_{1}, M_{2}, \cdots$ be the MVPDs in the start of each iteration. We know wty $\left(M_{k}\right)$ is obtained from wty $\left(M_{k-1}\right)$ by turning an $(r, c)$ into $(r, c-1)$. Thus, the algorithm must terminate.

Example 5.9. The following is an example of the algorithm. We start with a saturated $M \in \operatorname{MVPD}(w) \backslash \operatorname{MVPD}(w)$ where $w^{-1}$ has one-line notation 14253 . We first apply droop ${ }_{(1,1)}^{\prime}$ and obtain another $M_{2} \in \operatorname{MVPD}(w)$. Notice that $M_{2}$ is also saturated and $\mathrm{wt}_{M}(\mathbf{x})=\mathrm{wt}_{M_{2}}(\mathbf{x})$. We then apply droop ${ }_{(2,2)}^{\prime}$ and obtain $M^{\prime}$. Notice that $\mathrm{wt}_{M^{\prime}}(\mathbf{x})=\mathrm{wt}_{M}(\mathbf{x}) x_{3}$.


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