Top degree components of Grothendieck and Lascoux polynomials

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Outline

1. Grothendieck polynomials. Bottom layer and top layer.

2. Lascoux polynomials. Bottom layer and top layer.

3. Span of the top layers. Connections to a q-analogue of Bell numbers.

Grothendieck polynomials

Define $\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}}$, where s_i swaps x_i and x_{i+1}. For instance,

$$\partial_1(x_1^3 x_3) = \frac{x_1^3 x_3 - x_2^3 x_3}{x_1 - x_2} = x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3.$$

Then for $w \in S_n$,

$$\mathfrak{G}_w := \begin{cases} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} & \text{if } w = [n, n-1, \ldots, 1] \\ \partial_i((1 + x_{i+1})\mathfrak{G}_{ws_i}) & \text{if } w(i) < w(i+1). \end{cases}$$

Grothendieck polynomials

$$\begin{aligned}\mathfrak{G}_{2143} &= x_1^2 x_2 x_3 \\ &\quad + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 \\ &\quad + x_1^2 + x_1 x_2 + x_1 x_3\end{aligned}$$

- Inhomogeneous.
- Only have positive integer coefficients.

Schubert polynomials

$$\begin{aligned}\mathfrak{S}_{2143} &= x_1^2x_2x_3 \\ &+ x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 \\ &+ \color{red}{x_1^2 + x_1x_2 + x_1x_3} \qquad \leftarrow \text{Schubert polynomial } \mathfrak{S}_{2143}\end{aligned}$$

What is the leading monomial of a Schubert polynomial $\mathfrak{S}_w$?

Leading Monomial

Tail-Lex order: Compare monomials by first comparing the power of x_n, then x_{n-1}, x_{n-2}, $\ldots$.

$$x_1 x_2^3 x_3^2 > x_1^4 x_2 x_3^2$$

The **leading monomial** of a polynomial is the largest monomial in it.

$$\mathfrak{S}_{2143} = x_1^2 + x_1 x_2 + x_1 x_3$$

What is the leading monomial of $\mathfrak{S}_w$?

Inversion code

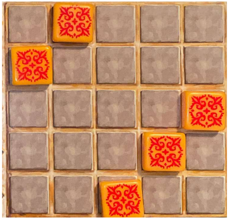
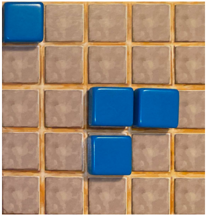
For $w \in S_n$, an **inversion** is (i, j) with $w(i) > w(j)$ and $i < j$.
The **inversion code** of w, denoted as $\mathrm{invcode}(w)$, is a code where the i^{th} entry is the number of inversions (i, j).
For instance, $\mathrm{invcode}(21543) = (1, 0, 2, 1, 0)$.

Theorem (Billey–Jockusch–Stanley)

The leading monomial of $\mathfrak{S}_w$ is $x^{\mathrm{invcode}(w)}$.

Rothe diagrams

Construct the Rothe diagram $RD(21543)$.

The weight of a diagram is a sequence where the i^{th} entry is the number of tiles on row i. $\mathrm{wt}(RD(21543)) = (1, 0, 2, 1, 0)$.

Fact: $\mathrm{invcode}(w) = \mathrm{wt}(RD(w))$.

Top layer of Grothendieck polynomials

$$\begin{aligned}\mathfrak{G}_{2143} = {} & x_1^2 x_2 x_3 && \leftarrow \widehat{\mathfrak{G}}_{2143} \\ & + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 \\ & + x_1^2 + x_1 x_2 + x_1 x_3\end{aligned}$$

How to compute their leading monomials?

rajcode

Definition (Pechenik–Speyer–Weigandt)

Take $w \in S_n$. For each i, find a longest increasing subsequence in w that starts at $w(i)$. Count how many numbers on the right of $w(i)$ are not involved.

- When $i = 1$, 21543. Three numbers not involved: 1,4,3.
- When $i = 2$, 21543. Two numbers not involved: 4,3.
- When $i = 3$, 21543. Two numbers not involved: 4,3.
- When $i = 4$, 21543. One number not involved: 3.
- When $i = 5$, 21543. No numbers not involved.

$$\mathrm{rajcode}(21543) = (3, 2, 2, 1, 0)$$

Leading monomial of $\widehat{\mathfrak{G}}_w$

Theorem (Pechenik–Speyer–Weigandt)

- *The leading monomial of* $\widehat{\mathfrak{G}}_w$ *is* $x^{\mathrm{rajcode}(w)}$.
- *Two permutations* u *and* v *have the same* rajcode *if and only if* $\widehat{\mathfrak{G}}_u$ *is a scalar multiple of* $\widehat{\mathfrak{G}}_v$.

For example, $\widehat{\mathfrak{G}}_{21543}$ has leading monomial $x_1^3 x_2^2 x_3^2 x_4$.

Lascoux polynomials

For weak compositions α,

$$\mathfrak{L}_\alpha := \begin{cases} x^\alpha & \text{if } \alpha_1 \geqslant \alpha_2 \geqslant \cdots \\ \partial_i(x_i(1+x_{i+1})\mathfrak{L}_{s_i\alpha}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

$$\mathfrak{G}_w := \begin{cases} x_1^{n-1}x_2^{n-2}\cdots x_{n-1} & \text{if } w = [n, n-1, \ldots, 1] \\ \partial_i((1+x_{i+1})\mathfrak{G}_{ws_i}) & \text{if } w(i) < w(i+1). \end{cases}$$

Theorem (Shimozono–Y)

For $w \in S_n$, $\mathfrak{G}_w$ expands **positively** *into $\mathfrak{L}_\alpha$.*

$$\mathfrak{G}_{2143} = \mathfrak{L}_{101} + \mathfrak{L}_2 + \mathfrak{L}_{201}$$

Key polynomials

$$\begin{aligned}\mathfrak{L}_{102} &= (x_1^2x_2^2x_3 + x_1^2x_2x_3^2) \\ &+ (x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1^2x_3^2) \\ &+ ({\color{red}x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_3^2}) \qquad \leftarrow \kappa_{102}\end{aligned}$$

Theorem (Lascoux–Schützenberger)

The key polynomial κ_α has leading monomial x^α.

Top layer of Lascoux polynomials

$$\begin{aligned}\mathfrak{L}_{102} &= (x_1^2x_2^2x_3 + x_1^2x_2x_3^2) \qquad\qquad \leftarrow \widehat{\mathfrak{L}}_{102}\\ &+ (x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1^2x_3^2)\\ &+ (x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_3^2)\end{aligned}$$

Question: What is the leading monomial of $\widehat{\mathfrak{L}}_\alpha$?

Key diagram

The following is the key diagram of $(1, 2, 0, 5, 2)$.

$D((1, 2, 0, 5, 2))$

Dark clouds

- Iterate through each row from bottom to top.
- For each row, find the rightmost tile on this row with no dark clouds underneath.
- If such a tile exists, turn it into a dark cloud.

$\longrightarrow$

Snow

Fill spaces **above** each dark cloud by snowflakes

$\longrightarrow$

rajcode of a weak composition

snow$(D((1,2,0,5,2)))$

Definition (Pan–Y)

The rajcode of a weak composition α *is* $\mathrm{wt}(\text{snow}(D(\alpha)))$.

For instance, $\text{rajcode}((1,2,0,5,2)) = (3,3,2,5,2)$

Top Lascoux polynomials

Theorem (Pan–Y)

- *The leading monomial of* $\widehat{\mathfrak{L}}_\alpha$ *is* $x^{\mathsf{rajcode}(\alpha)}$.
- *Two weak compositions* α *and* γ *have the same* rajcode *if and only if* $\widehat{\mathfrak{L}}_\alpha$ *is a scalar multiple of* $\widehat{\mathfrak{L}}_\gamma$.

Theorem (Pechenik–Speyer–Weigandt)

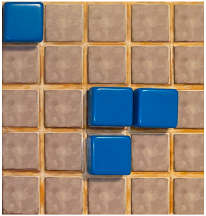
- *The leading monomial of* $\widehat{\mathfrak{G}}_w$ *is* $x^{\mathsf{rajcode}(w)}$.
- *Two permutations* u *and* v *have the same* rajcode *if and only if* $\widehat{\mathfrak{G}}_u$ *is a scalar multiple of* $\widehat{\mathfrak{G}}_v$.

Corollary (Pan–Y)

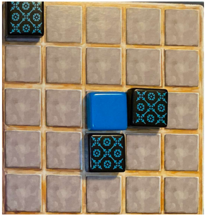
Let raj(α) *be the sum of entries in* rajcode(α). *Then* $\widehat{\mathfrak{L}}_\alpha$ *has degree* raj(α).

Snow on Rothe diagrams

We can do the same construction on a Rothe diagram.



$\rightarrow$



$\rightarrow$

$RD(21543)$ snow$(RD(21543))$

Theorem (Pan–Y)

The weight of snow$(RD(w))$ *is the same as* rajcode(w) *defined by Pechenik, Speyer and Weigandt.*

Space spanned by $\widehat{\mathfrak{G}}_w$

Let $\widehat{V_n}$ be the $\mathbb{Q}$-span of $\widehat{\mathfrak{G}}_w$ with $w \in S_n$.

Proposition (Pan–Y)

Let C_n be the set of weak compositions entry-wise at most $(n-1, \cdots, 2, 1, 0)$. Then $\widehat{V_n}$ is also the $\mathbb{Q}$-span of $\widehat{\mathfrak{L}}_\alpha$ with $\alpha \in C_n$.

Questions:

- Can we find bases of $\widehat{V_n}$? A basis consisting of $\widehat{\mathfrak{G}}_w$ by [Pechenik–Speyer–Weigandt]
- What is the dimension of $\widehat{V_n}$? B_n, the n^{th} Bell number by [Pechenik–Speyer–Weigandt]
- What is the Hilbert series of $\widehat{V_n}$?

Extracting a basis from a spanning set

Recall:

Theorem (Pan–Y)

- *The leading monomial of $\widehat{\mathfrak{L}}_\alpha$ is $x^{\mathrm{rajcode}(\alpha)}$.*
- *Two weak compositions α and γ have the same* rajcode *if and only if $\widehat{\mathfrak{L}}_\alpha$ is a scalar multiple of $\widehat{\mathfrak{L}}_\gamma$.*

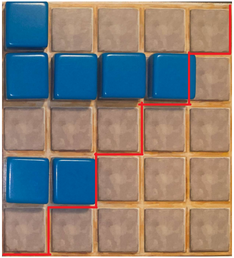
Partition the spanning set of $\widehat{V_4}$ into equivalence classes by rajcode:

$$\{0000\}, \{1000\}, \{0100, 1100\},$$
$$\{0010, 1010, 0110, 1110\},$$
$$\{2000\}, \{0200\}, \{1200, 2200\},$$
$$\{2100\}, \{2010, 2110\}, \{3000\},$$
$$\{0210, 1210, 2210\}, \{3100\},$$
$$\{3200\}, \{3010, 3110\}, \{3210\}.$$

Snowy weak compositions

Definition (Pan–Y)

A weak composition is **snowy** *if its positive entries are distinct.*



a snowy weak composition in C_6

Theorem (Pan–Y)

Each $\alpha \in C_n$ has the same rajcode as exactly one snowy weak composition in C_n.

Bases of $\widehat{V_n}$

Theorem (Pan–Y)

The space $\widehat{V_n}$ has basis $\{\widehat{\mathfrak{L}}_\alpha : \alpha \in C_n \textit{ is snowy}\}$.

For instance, $\widehat{V_3}$ has basis

$$\{\widehat{\mathfrak{L}}_{000}, \widehat{\mathfrak{L}}_{100}, \widehat{\mathfrak{L}}_{010}, \widehat{\mathfrak{L}}_{200}, \widehat{\mathfrak{L}}_{210}, \cancel{\widehat{\mathfrak{L}}_{110}}\},$$

and $\widehat{V_4}$ has basis

$$\begin{aligned}\{&\widehat{\mathfrak{L}}_{0000}, \widehat{\mathfrak{L}}_{0100}, \widehat{\mathfrak{L}}_{0010}, \widehat{\mathfrak{L}}_{0200}, \widehat{\mathfrak{L}}_{0210}, \cancel{\widehat{\mathfrak{L}}_{0110}},\\ &\widehat{\mathfrak{L}}_{1000}, \cancel{\widehat{\mathfrak{L}}_{1100}}, \cancel{\widehat{\mathfrak{L}}_{1010}}, \widehat{\mathfrak{L}}_{1200}, \cancel{\widehat{\mathfrak{L}}_{1210}}, \cancel{\widehat{\mathfrak{L}}_{1110}},\\ &\widehat{\mathfrak{L}}_{2000}, \widehat{\mathfrak{L}}_{2100}, \widehat{\mathfrak{L}}_{2010}, \cancel{\widehat{\mathfrak{L}}_{2200}}, \cancel{\widehat{\mathfrak{L}}_{2210}}, \cancel{\widehat{\mathfrak{L}}_{2110}},\\ &\widehat{\mathfrak{L}}_{3000}, \widehat{\mathfrak{L}}_{3100}, \widehat{\mathfrak{L}}_{3010}, \widehat{\mathfrak{L}}_{3200}, \widehat{\mathfrak{L}}_{3210}, \cancel{\widehat{\mathfrak{L}}_{3110}}\}.\end{aligned}$$

Hilbert series

Definition

Suppose A is a polynomial vector space with a basis $\mathfrak{B}$ consisting of homogeneous polynomials. Then $\mathrm{Hilb}(A; q) := \sum_{f \in \mathfrak{B}} q^{\deg(f)}$.

For instance, $\widehat{V_3}$ has basis

$$\{\widehat{\mathfrak{L}}_{000}, \widehat{\mathfrak{L}}_{100}, \widehat{\mathfrak{L}}_{010}, \widehat{\mathfrak{L}}_{200}, \widehat{\mathfrak{L}}_{210}\}, \text{ so}$$

$$\mathrm{Hilb}(\widehat{V_3}; q) = q^0 + q^1 + q^2 + q^2 + q^3.$$

In general,

$$\mathrm{Hilb}(\widehat{V_n}; q) = \sum_{\text{snowy } \alpha \in C_n} q^{raj(\alpha)},$$

where raj(α) is the sum of rajcode(α).

q-analogue of Bell numbers

Define the polynomial $S_{n,k}(q)$ recursively:

$$S_{n+1,k}(q) = q^{k-1}S_{n,k-1}(q) + [k]_q S_{n,k}(q),$$

with base cases $S_{0,k}(q) = S_{0,k}$. It is called a q-**analogue of** $S_{n,k}$.

The q-**analogue of** B_n is $B_n(q) := \sum_{k=0}^{n} S_{n,k}(q)$.

Question: How to write $B_n(q)$ as a generating function?

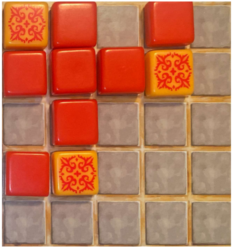
Non-attacking rook diagrams

A **non-attacking rook diagram** is a diagram with at most one tile in each column or row.
Let Rook_n be the set of non-attacking rook diagrams within the staircase that has length $n - r$ in row r.

An element of Rook_6

$B_n(q)$ via Rooks

Define the Northwest statistic $\mathsf{NW}(\cdot)$ on Rook_n.



$$\mathsf{NW}(R) = 10$$

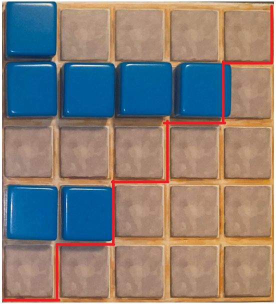
Theorem (Garsia–Remmel)

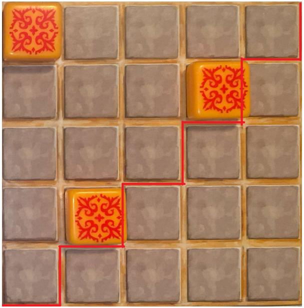
$$\sum_{R \in \mathsf{Rook}_n} q^{\mathsf{NW}(R)} = \mathrm{rev}(B_n(q)),$$

where $\mathrm{rev}(\cdot)$ *means to reverse all the coefficients.*

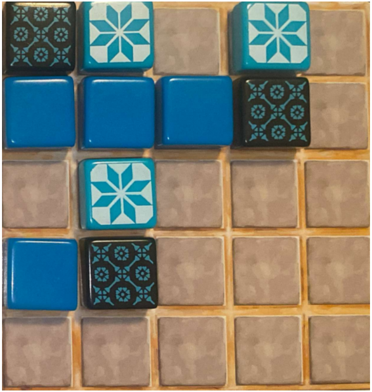
Bijection

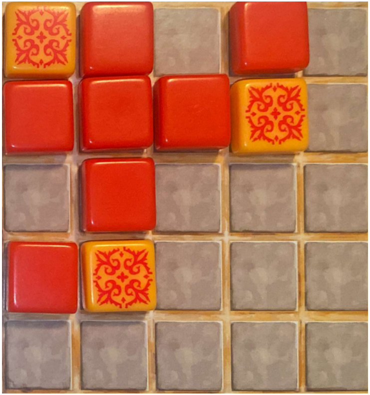
There is a bijection between snowy weak compositions in C_n and Rook_n.





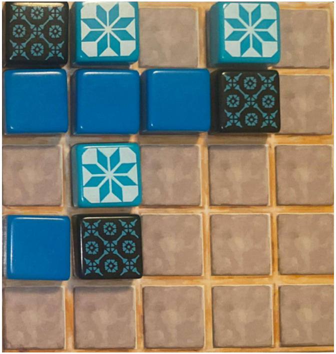
Bijection





If $\alpha \mapsto R$, then $\mathrm{raj}(\alpha) = \mathrm{NW}(R)$.

Hilbert series of $\widehat{V_n}$



$$\mathrm{Hilb}(\widehat{V_n}; q) = \sum_{\substack{\alpha \in C_n, \\ \alpha \text{ is snowy}}} q^{\mathrm{raj}(\alpha)} = \sum_{R \in \mathrm{Rook}_n} q^{\mathrm{NW}(R)} = \mathrm{rev}(B_n(q))$$

Span of all top Lascoux polynomials

Let $\widehat{V}$ be the span of all top Lascoux polynomials.

Theorem (Pan–Y)

- $\widehat{V_1} \subseteq \widehat{V_2} \subseteq \cdots \subseteq \widehat{V}$.
- $\widehat{V} = \bigcup_{n \geq 1} \widehat{V_n}$.
- $\widehat{V}$ *has basis* $\{\widehat{\mathfrak{L}}_\alpha : \alpha \textit{ is snowy}\}$.
- $\mathrm{Hilb}(\widehat{V}; q) = \lim_{n \to \infty} \mathrm{Hilb}(\widehat{V_n}; q) = \prod_{m>0}(1 + \frac{q^m}{1-q})$

$\widehat{V}$ is a ring

$$\widehat{\mathfrak{L}}_{(0,3,1,5)} \times \widehat{\mathfrak{L}}_{(2,4,0,0)}$$
$$=\widehat{\mathfrak{L}}_{(4,8,1,5)} + \widehat{\mathfrak{L}}_{(6,7,1,5)} + \widehat{\mathfrak{L}}_{(5,9,1,4)} + 2\widehat{\mathfrak{L}}_{(6,8,1,4)} + \widehat{\mathfrak{L}}_{(6,9,1,3)} + \widehat{\mathfrak{L}}_{(7,8,1,3)}.$$

Theorem (Y 2023+)

If α and β are snowy, then $\widehat{\mathfrak{L}}_\alpha \times \widehat{\mathfrak{L}}_\beta$ can be expanded into top Lascoux polynomials with positive coefficients.

Problem: Find a combinatorial formula for the coefficients.

Relations between $\widehat{\mathfrak{L}}_\alpha$ and the Schubert polynomials

Every $\widehat{\mathfrak{L}}_\alpha$ is a Schubert polynomial, after "reversal".

$$\widehat{\mathfrak{L}}_{31524} = x^{(5,5,5,3,3)} + x^{(5,5,4,4,3)} + x^{(5,4,5,4,3)} + x^{(5,5,4,3,4)} + x^{(5,4,5,3,4)}$$

$$\mathfrak{S}_{24153} = x^{(2,2,0,0,0)} + x^{(2,1,1,0,0)} + x^{(2,1,0,1,0)} + x^{(1,2,1,0,0)} + x^{(1,2,0,1,0)}$$

A solution of this problem would solve the Schubert problem.

Thank you!