Homework 1 Math180C Fall 2019

Due Wednesday Oct 9, 11:59pm, please submit on Gradescope. Relevant sections in the textbook: 6.1, 6.2, 6.3. Justify all your answers.

1. A pure birth process starting from $X(0) = 0$, with birth rates $(\lambda_k)_{k=0}^{\infty}$ where $\lambda_k = 1 + k$. Let W_4 be the time it takes the process to reach state 4. Compute the mean and variance of W_4.

2. Let $X(t)$ be a pure birth process with birth rates $\lambda_0 = 2$, $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 5, \ldots$, starting from $X(0) = 0$. Compute the transition probabilities $p_{0,t}(0,0)$, $p_{0,t}(0,1)$. Recall that the definition of transition probability is

$$p_{s,t}(i,j) = \mathbb{P}(X(t) = j | X(s) = i).$$

3. Consider a population with a fixed number N of individuals. Suppose at time $t = 0$ there is exactly one infected individual and $N-1$ uninfected individuals. Once infected, the individual stays infected forever. Any given infected individual transmit the disease to any given susceptible individual at rate α. Denote by $X(t)$ the number of infected individuals in the population at time t.

(i) Explain that $(X(t), t \geq 0)$ is a pure birth process and specify the birth parameters.

(ii) Observe that N is an absorbing state (all individuals infected). Compute the expected time it takes the process to be absorbed in N.

4. Consider a pure birth process with birth rates

$$\lambda_k = (1+k)^{\rho}, \text{ for } k = 1, 2, ...,$$

where ρ is a positive constant. The process starts at $X(0) = 1$. Determine for what values of ρ the process explodes in finite time.

5. Determine the transition probabilities for the pure death process described by $X(0) = 3$, $\mu_3 = 1$, $\mu_2 = 2$ and $\mu_1 = 3$. That is, give the transition probabilities $p_{s,t}(i,j)$ explicitly for $s < t$, $i, j \in \{0, 1, 2, 3\}$.

6. Let $X(t)$ be a pure death process with constant death rates $\mu_k = \theta$ for $k = 1, 2, \ldots, N$. Determine $p_{0,t}(N, j)$ for $j = 0, 1, \ldots, N$. (Hint: think about the connection with the Poisson process.)

7. Let T be the time to extinction in the pure death process with death rates $\mu_k = k\alpha$, $k = 1, \ldots, N$, starting at $X(0) = N$. Determine the mean and variance of T.

8. Particles are emitted by a radioactive substance according to a Poisson process of rate λ. Each particle exists for an exponentially distributed length of time, independent of other particles, before disappearing. Let $X(t)$ denote the number of particles alive at time t. Suppose that $X(0) = 1$.

(i) Explain that $X(t)$ is a birth and death process and determine the parameters. (Pay attention to the special state 0.)

(ii) Compute the probability that starting at 1, the process $X(t)$ gets to 0 before 3.