Homework 2 Math180C Fall 2019

Due Wednesday Oct 16, 11:59pm, please submit on Gradescope. Relevant sections in the textbook: 6.3, 6.5.1, 6.6. Justify all your answers.

1. (Moran model with selection) Consider a population of N individuals of type A and B. Each type A individual independently lives for an Exponential(1) time, and each type B individual lives independently for Exponential$(1-s)$ time, $0 < s < 1$; then gets replaced by a new individual whose parent is chosen uniformly random from the population. Let $X(t)$ be the number of B individuals at time t.

(i) Explain that $(X(t), t \geq 0)$ is a continuous time birth and death chain, and determine the birth rates $(\lambda_k)_{k=0}^{\infty}$ and the death rates $(\mu_k)_{k=1}^{\infty}$.

(ii) Calcalute the probability that starting with $X(0) = 1$, the process hits N before hitting 0, that is $\mathbb{P}(\tau_N < \tau_0 | X(0) = 1)$.

2. Let $(X(t), t \geq 0)$ be a continuous time birth and death chain on $\{0, 1, 2, \ldots\}$, $X(0) = 1$ with birth rate $\lambda_k = k\lambda$ and death rate $\mu_k = k\mu$, $k \in \mathbb{N}$. Calculate $\mathbb{P}(\tau_0 < \infty | X(0) = 1)$, where $\tau_0 = \inf\{t \geq 0 | X(t) = 0\}$.

3. Consider a birth and death process on $\{0, 1, 2, 3, 4\}$ with birth rates $\lambda_0 = 0$, $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$ and $\lambda_4 = 0$; and death rates $\mu_0 = 0$, $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 2$ and $\mu_4 = 0$. Note that 0 and 4 are absorbing states. Suppose the process starts at $X(0) = 2$,compute the expected time to absorption.

4. Let $(X_1(t), t \geq 0)$ and $(X_2(t), t \geq 0)$ be two independent continuous Markov chains on two states $\{0, 1\}$, with transition rates $q(0, 1) = \alpha$, $q(1, 0) = \beta$ (the two processes have the same transition rates). Explain that $(X_1(t) + X_2(t), t \geq 0)$ is a continuous time Markov chain and determine its state space and transition rates.

5. Let $(Y_n)_{n=0}^{\infty}$ be a discrete time Markov chain on state space $\{1, 2, \ldots, n\}$ with transition matrix $P = (P_{i,j})_{1 \leq i,j \leq n}$. Let $(N(t), t \geq 0)$ be a Poisson process of rate λ, independent of the Markov chain $(Y_n)_{n=0}^{\infty}$. Consider the process defined by

$$X(t) = Y_{N(t)}.$$

Explain that $(X(t), t \geq 0)$ is a continuous time Markov chain and determine its transition rates.

6. Consider a two state continuous time Markov chain on $\{1, 2\}$, with transition rates $q(1, 2) = \lambda$, $q(2, 1) = \mu$. Calculate transition probabilities $p_t(1, 2)$, $p_t(2, 1)$.