Homework 2 Math280B Winter 2018

Due Friday in class, Feb 16. Relevant sections in Durrett's textbook 3.3, 3.4, 3.6 and 5.1. Justify all your answers.

Throughout this homework, Z denotes a random variable that has a standard normal distribution $\mathcal{N}(0,1)$.

1. Suppose $X_1, X_2, \dots$ are i.i.d. random variables such that $\mathbb{E}[X_m] = 0$ and $E[X_m^2] = \sigma^2$ for all m, where $0 < \sigma^2 < \infty$. Show that

$$\sum_{m=1}^{n} X_m \Big/ \left(\sum_{m=1}^{n} X_m^2\right)^{1/2} \Rightarrow Z.$$

2. For $n \in \mathbb{N}$, suppose X_n has the binomial distribution with parameters n and p_n. This means that $X_n = \xi_{n,1} + \ldots + \xi_{n,n}$, where $\xi_{n,j}$, $1 \le j \le n$, are i.i.d. and $\mathbb{P}(\xi_{n,i} = 1) = p_n$ and $\mathbb{P}(\xi_{n,i} = 0) = 1 - p_n$.

(a) Show that if $\lim_{n\to\infty} np_n(1-p_n) = \infty$, then

$$\frac{X_n - np_n}{\sqrt{np_n(1-p_n)}} \Rightarrow Z.$$

(b) Show that if $\lim_{n\to\infty} np_n = \lambda \in (0,\infty)$, then $X_n \Rightarrow \text{Poisson}(\lambda)$.

3. Suppose $X_1, X_2, \ldots$ are independent. Suppose $\mathbb{P}(X_m = -m) = \mathbb{P}(X_m = m) = m^{-2}/2$ for all $m \ge 1$ and $\mathbb{P}(X_m = -1) = \mathbb{P}(X_m = 1) = (1 - m^{-2})/2$ for $m \ge 2$. Let $S_n = X_1 + \ldots + X_n$. Show that $\text{var}(S_n)/n \to 2$ but $S_n/\sqrt{n} \Rightarrow Z$.

4. Suppose X and Y are random variables such that $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$, and suppose $\mathcal{G}$ is a σ-field. Show that if $\mathbb{E}[Y|\mathcal{G}] = X$ and $\mathbb{E}[Y^2|\mathcal{G}] = X^2$, then $X = Y$ a.s.

5. If $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{G}$ are σ-fields, we say $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent given $\mathcal{G}$ if $\mathbb{P}(A \cap B|\mathcal{G}) = \mathbb{P}(A|\mathcal{G})\mathbb{P}(B|\mathcal{G})$ for all $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Prove that if $\mathbb{P}(A|\sigma(\mathcal{F}_2, \mathcal{G})) = \mathbb{P}(A|\mathcal{G})$ for all $A \in \mathcal{F}_1$, where $\sigma(\mathcal{F}_2, \mathcal{G})$ denotes the σ-field generated by $\mathcal{F}_2$ and $\mathcal{G}$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent given $\mathcal{G}$.

6. Suppose $f : \mathbb{R}^2 \to [0,\infty)$ is a measurable function, and X and Y are random variables with joint density f . Let $g(x) = \int_{\mathbb{R}} f(x,y)dy$, and for simplicity assume $g(x) > 0$ for all $x \in \mathbb{R}$. Let $h(x,y) = f(x,y)/g(x)$. Now for $\omega \in \Omega$ and $B \in \mathcal{B}(\mathbb{R})$, let

$$Q(\omega, B) = \int_B h(X(\omega), y)dy.$$

(a) Show that g is a density for X.

(b) Show that Q is a regular conditional distribution for Y given $\sigma(X)$.