Due Friday in class, Feb 16. Relevant sections in Durrett’s textbook 3.3, 3.4, 3.6 and 5.1. Justify all your answers.

Throughout this homework, Z denotes a random variable that has a standard normal distribution N(0,1).

1. Suppose \( X_1, X_2, \ldots \) are i.i.d. random variables such that \( \mathbb{E}[X_m] = 0 \) and \( \mathbb{E}[X_m^2] = \sigma^2 \) for all \( m \), where \( 0 < \sigma^2 < \infty \). Show that

\[
\sum_{m=1}^{n} X_m / \left( \sum_{m=1}^{n} X_m^2 \right)^{1/2} \Rightarrow Z.
\]

2. For \( n \in \mathbb{N} \), suppose \( X_n \) has the binomial distribution with parameters \( n \) and \( p_n \). This means that \( X_n = \xi_{n,1} + \ldots + \xi_{n,n} \), where \( \xi_{n,j} \), \( 1 \leq j \leq n \), are i.i.d. and \( \mathbb{P}(\xi_{n,i} = 1) = p_n \) and \( \mathbb{P}(\xi_{n,i} = 0) = 1 - p_n \).

(a) Show that if \( \lim_{n \to \infty} np_n(1 - p_n) = \infty \), then

\[
\frac{X_n - np_n}{\sqrt{np_n(1-p_n)}} \Rightarrow Z.
\]

(b) Show that if \( \lim_{n \to \infty} np_n(1 - p_n) = \lambda \in (0, \infty) \), then \( X_n \Rightarrow \text{Poisson}(\lambda) \).

3. Suppose \( X_1, X_2, \ldots \) are independent. Suppose \( \mathbb{P}(X_m = -m) = \mathbb{P}(X_m = m) = m^{-2}/2 \) for all \( m \geq 1 \) and \( \mathbb{P}(X_m = -1) = \mathbb{P}(X_m = 1) = (1-m^{-2})/2 \) for \( m \geq 2 \). Let \( S_n = X_1 + \ldots + X_n \).

Show that \( \text{var}(S_n)/n \to 2 \) but \( S_n/\sqrt{n} \nrightarrow Z \).

4. Suppose \( X \) and \( Y \) are random variables such that \( \mathbb{E}[X^2] < \infty \) and \( \mathbb{E}[Y^2] < \infty \), and suppose \( \mathcal{G} \) is a \( \sigma \)-field. Show that if \( \mathbb{E}[Y|\mathcal{G}] = X \) and \( \mathbb{E}[Y^2|\mathcal{G}] = X^2 \), then \( X = Y \) a.s.

5. If \( \mathcal{F}_1, \mathcal{F}_2, \) and \( \mathcal{G} \) are \( \sigma \)-fields, we say \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are conditionally independent given \( \mathcal{G} \) if \( \mathbb{P}(A \cap B|\mathcal{G}) = \mathbb{P}(A|\mathcal{G})\mathbb{P}(B|\mathcal{G}) \) for all \( A \in \mathcal{F}_1 \) and \( B \in \mathcal{F}_2 \). Prove that if \( \mathbb{P}(A|\sigma(\mathcal{F}_2, \mathcal{G})) = \mathbb{P}(A|\mathcal{G}) \) for all \( A \in \mathcal{F}_1 \), where \( \sigma(\mathcal{F}_2, \mathcal{G}) \) denotes the \( \sigma \)-field generated by \( \mathcal{F}_2 \) and \( \mathcal{G} \), then \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are conditionally independent given \( \mathcal{G} \).

6. Suppose \( f : \mathbb{R}^2 \to [0, \infty) \) is a measurable function, and \( X \) and \( Y \) are random variables with joint density \( f \). Let \( g(x) = \int_{\mathbb{R}} f(x, y) \, dy \), and for simplicity assume \( g(x) > 0 \) for all \( x \in \mathbb{R} \). Let \( h(x, y) = f(x, y)/g(x) \).

Now for \( \omega \in \Omega \) and \( B \in \mathcal{B}(\mathbb{R}) \), let

\[
Q(\omega, B) = \int_{B} h(X(\omega), y) \, dy.
\]
(a) Show that $g$ is a density for $X$.

(b) Show that $Q$ is a regular conditional distribution for $Y$ given $\sigma(X)$. 