Homework 3 Math280B Winter 2018

Due Friday in class, March 9. Relevant sections in Durrett's textbook 5.2, 5.4, 5.5 and 5.7. Justify all your answers.

1. Suppose $(X_n)_{n=0}^{\infty}$ and $(Y_n)_{n=0}^{\infty}$ are martingales with respect to filtration $(\mathcal{F}_n)_{n=0}^{\infty}$. Assume $\mathbb{E}[X_n^2] < \infty$ and $\mathbb{E}[Y_n^2] < \infty$ for all n.

(a) Show that for all n,

$$\mathbb{E}[X_n Y_n - X_0 Y_0] = \sum_{m=1}^{n} \mathbb{E}\left[(X_m - X_{m-1})(Y_m - Y_{m-1})\right].$$

(b) Show that if $X_0 = 0$, then for all n,

$$\mathbb{E}[X_n^2] = \sum_{m=1}^{n} \mathbb{E}[(X_m - X_{m-1})^2].$$

2. Let $(S_n)_{n=0}^{\infty}$ be a simple random walk. That is, $S_0 = 0$ and, for $n \geq 1$, we have $S_n = \xi_1 + \ldots + \xi_n$, where $\xi_1, \xi_2, \ldots$ are i.i.d. with $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = 1/2$.

(a) Show that $S_n^2 - n$ is a martingale.

(b) Let $T = \inf\{n : \ S_n \notin (-a, a)\}$,where $a \in \mathbb{N}$. Show that $\mathbb{E}[T] = a^2$.

3. Let $a > 0$, and let $X_1, X_2, ...$ be i.i.d. random variables having a normal distribution with mean $\mu > 0$ and variance 1. Let $S_0 = a$, and let $S_n = a + X_1 + \cdots + X_n$ for $n \in \mathbb{N}$.

(a) Let $Y_n = e^{-2\mu S_n}$. Show that $(Y_n)_{n=0}^{\infty}$ is a martingale.

(b) Show that

$$\mathbb{P}\left(S_n \leq 0 \text{ for some } n\right) \leq e^{-2\mu a}.$$

4. Let $(X_n)_{n=0}^{\infty}$ be a submartingale such that $X_0 = 0$ and

$$\sup_{n \geq 0} X_n(\omega) < \infty$$

for all $\omega \in \Omega$. Let $\xi_n = X_n - X_{n-1}$, and suppose $\mathbb{E}[\sup_{n\geq 0} \xi_n^+] < \infty$. Show that $(X_n)_{n=0}^{\infty}$ converges a.s.

5. Let $(X_n)_{n=0}^{\infty}$ be a martingale such that $X_0 = 0$ and $|X_{n+1} - X_n| \leq r$ for all n, where r is a positive real number.

(a) Show that

$$\mathbb{E}\left[\max_{0\le m\le n} X_m^2\right] \le 4r^2 n.$$

(b) Show that if $x > 0$, then

$$\mathbb{P}\left(\max_{0\le m\le n} |X_m| > x\sqrt{n}\right) \le \frac{r^2}{x^2}.$$

6. Let $(\mathcal{F}_n)_{n=0}^{\infty}$ be a filtration, and let $\mathcal{F}_\infty = \sigma(\cup_{n=0}^{\infty}\mathcal{F}_n)$. Let A be an event such that $A \in \mathcal{F}_\infty$ but A is independent of $\mathcal{F}_0$. Suppose $\mathbb{P}(A) = 1/2$. Let $X_n = \mathbb{P}(A|\mathcal{F}_n)$ for all $n \ge 0$.

(a) Show that if $\frac{1}{2} \le x \le 1$, then

$$\mathbb{P}\left(\sup_{n\ge 0} X_n \ge x\right) \le \frac{1}{2x}.$$

(b) Show that

$$\frac{3}{4} \le \mathbb{E}\left[\sup_{n\ge 0} X_n\right] \le \frac{1+\log 2}{2}.$$