Homework 2 Math280C Spring 2018

Due Friday in class, April 27. Relevant sections in Durrett's textbook 8.2, 8.3 and 8.5. Justify all your answers.

1. (was Problem 5 HW1) Let $(X_t)_{t\in\mathbb{R}}$ be an Ornstein-Uhlenbeck process as above. Show that $(X_t)_{t\geq 0}$ is a Markov process and write down its Markov transition kernels.

2. (Do the calculus to derive the arcsine law) Let $B(t)$ be standard Brownian motion, $T_0 = \inf\{s > 0, B_s = 0\}$ and $L = \sup\{t \leq 1 : B_t = 0\}$. Deduce from the Markov property that

$$\mathbb{P}(L \leq t) = \int \mathbb{P}_y(T_0 > 1 - t)p(t, 0, dy),$$

where $p(t, x, \cdot)$ is the Markov transition kernel of Brownian motion. Determine the law of L explicitly.

3. Let L be as in Problem 2. Show that a.s. there exists times $t_n < s_n < L$ with $t_n \uparrow L$ such that $B(t_n) < 0$ and $B(s_n) > 0$ for all n.

4. Consider the 2-dimensional Brownian motion on $\mathbb{R}^2$, $B(t) = (B_1(t), B_2(t))$, where B_1, B_2 are two independent standard Brownian motions. We inerpret $\mathbb{R}^2$ as the complex plane, that is identify $B(t)$ as $B_1(t)+iB_2(t)$. A complex valued process is called a martingale if both its real and imaginary parts are martingales.

(i) Show that for any $\lambda \in \mathbb{R}$, $\left(e^{i\lambda B(t)}\right)_{t\geq 0}$ is a martingale.

(ii) Suppose $B(t)$ starts at i, that is $B_1(0) = 0$, $B_2(0) = 1$. Let T be the first time when $B(t)$ hits the real axis. Show that

$$\mathbb{E}\left[e^{i\lambda B(T)}\right] = e^{-\lambda}.$$

(Note that this exactly tells you what is the ch.f. of $B(T)$, by the inversion formula we have that $B(T)$ has Cauchy distribution.)

5. Let T_x be the first hitting time of a point $x \in \mathbb{R}$. Let $R > 0$, τ be the first hitting time of the set $\{-R, R\}$. Consider a Brownian motion started at $x \in (0, R)$. Calculate

$$\mathbb{E}_x[\tau],\ \mathbb{E}_x[T_R | T_R < T_0].$$