Homework 3 Math280C Spring 2018

Due Friday in class, May 11. Relevant sections in Durrett's textbook 8.5 and 8.6.; in Morters-Peres book 5.3 and 5.4. Justify all your answers.

1. Show that $(1+t)^{-1/2}\exp\left(\frac{B_t^2}{2(1+t)}\right)$ is a martingale. Use this martingale to show that

$$\limsup_{t\to\infty}\frac{B_t}{((1+t)\log(1+t))^{1/2}}\leq\frac{1}{\sqrt{2}}\ a.s.$$

2. Suppose that $(S_n)_{n\geq 0}$ is a random walk with mean 0, variance 1 increments. Define $\{S_n^*(t), t\in[0,1]\}$ as in the Donsker's invariance principle. Show that

$$m\left(\{t\in[0,1]:\ S_n^*(t)>0\}\right)-\frac{1}{n}\#\{k\in\{1,\dots,n\}:S_k>0\}$$

converges to 0 in probability, where m denotes the Lebesgue measure on $[0,1]$.

3. (Continued from Exercise 2) Suppose that $(S_n)_{n\geq 0}$ is a random walk with mean 0, variance 1 increments. Let $P_n=\#\{k\in\{1,\dots,n\}:S_k>0\}$. Show that P_n satsifies that for any $x\in(0,1)$,

$$\lim_{n\to\infty}\mathbb{P}\left(P_n\leq xn\right)=\frac{2}{\pi}\arcsin(\sqrt{x}).$$

4. (Doob H-transform) Let $P(x,y)$ be the transition matrix of an irreducible Markov chain $(X_n)_{n\geq 0}$ on a finite state space V. For $a,b\in V$, let $T_{a,b}$ be the hitting time $T_{a,b}=\min\{j\geq 0,\ X_j\in\{a,b\}\}$. Define

$$H(x)=\mathbb{P}_x\left(X_{T_{a,b}}=b\right).$$

Show that the chain $(X_n)_{n\geq 0}$ conditioned on reaching b before a and being absorbed at b has the same law as the Markov chain $(Y_j)_{j\geq 0}$ on $V\setminus\{a\}$ with transition probabilities

$$\hat{P}(x,y)=P(x,y)\frac{H(y)}{H(x)},\ \text{for } x\neq b.$$