Homework 4 Math280C Spring 2018

Due Friday in class, June 1. Relevant sections in Levin-Peres-Wilmer book: Ch. 1-4, Ch. 12, 15. Justify all your answers.

1. Suppose P is a Markov transition matrix on a finite state space S, P is irreducible and aperiodic. Define

$$\alpha_n = \sup_{i,j \in S} \left\| P^n(i, \cdot) - P^n(j, \cdot) \right\|_{TV}.$$

Show that $-\frac{1}{n} \log \alpha_n$ converges to a limit λ and moreover $\lambda > 0$.

2. Let $X_t = \left(X_t^1, \ldots, X_t^n\right)$ be the position of the lazy random walk on the hypercube $\{0,1\}^n$ (pick a coordinate uniformly at random, then randomize that digit to be uniform on $\{0,1\}$). Suppose $X_0 = (1, 1, \ldots, 1)$, that is the chain starts at all 1. Show that the covariance between X_t^i and X_t^j is negative. Conclude that $\mathrm{Var}\left(\sum_{i=1}^n X_t^i\right) \leq n/4$.

3. Let P be a Markov transition matrix reversible with respect to distribution π. Show that for any $t \in \mathbb{N}$, $P^{2t+2}(x,x) \leq P^{2t}(x,x)$.

4. Let (G_n) be a sequence of expander graphs with maximal degree Δ. Find a number $\beta(\Delta)$ such that for $\beta > \beta(\Delta)$, the relaxation time for Glauber dynamics for the Ising model grows exponentially in n. (This is Exercise 15.1 in the LPW book, relevant definitions can be found in that chapter.)

5. Take two $n \times n$ square grids $\{0, \ldots, n\}^2$ and $\{-n, \ldots, 0\}^2$ attached by identifying the two corners $(0,0)$. Denote by G the resulting graph, it has $2(n+1)^2 - 1$ vertices. Consider the transition kernel

$$P(x,y) = \begin{cases} 0 & \text{if } |x-y| > 1 \\ 1/4 & \text{if } |x-y| = 1 \\ 0 & \text{if } x = y \text{ is inside or } x = y = 0 \\ 1/4 & \text{if } x = y \text{ is on the boundary but not a corner} \\ 1/2 & \text{if } x = y \text{ is a corner.} \end{cases}$$

Show that P is reversible with respect to the uniform measure on G. Try to estimate the spectral gap of P from above and below. You don't have to match up your upper and lower bounds. This chain is Example 3.2.5 : What is the spectral gap of the dog (with no ears legs or tail for simplicity) in "*Lectures on finite Markov chains*" by Laurent Saloff-Coste.

Topics that we (plan to) cover in class:

- Chapter 12 LPW on eigenvalues and spectral exapnsion of reversible Markov chains.
- Chapter 13 (13.3-13.6) on Dirichlet forms and basic comparison
- Part of Chapter 15 on fast mixing of Glauber dynamics of high temperature Ising.
- Log-Sobolev inequalities and concentration of measure

Topics suggested for presentation:

- Around coupling, from Chapter 5, 14
- Shuffling cards, from Chapter 8
- Cover times, Chapter 11. In recent years there are marvelous new results on this topic.
- Introduction to cutoff phenomenon, possibly some examples explained, Chapter 18.

Any topic that interests you and related to Markov chains is welcome!