280C HW1 Idea

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1. We will show the strictly increasing part. The decreasing part is similar. Let

$$\begin{aligned} A &= \{\text{There is an interval } (a,b) \text{ such that } B_t \text{ is strictly increasing on } (a,b)\} \\ &= \{\text{There is an interval } (a,b) \text{ with rational endpoints such that } B_t \text{ is strictly increasing on } (a,b)\} \end{aligned}$$

by density of rational numbers. So it suffices to show $E = \{B_t \text{ is strictly increasing on } (a,b)\}\}$ has probaiblity 0. Let

$$E_n = \{B_{\frac{i+1}{n}(b-a)+a} - B_{\frac{i}{n}(b-a)+a>0}, \forall i = 0, \dots, n-1\}.$$

Then by independent increments of Brownian motion, $\mathbb{P}(E_n) = 2^{-n}$. Since $A \subseteq E_n$ for all n,

$$\mathbb{P}(A) \le \mathbb{P}(E_n) = 2^{-n}$$

for all n and $\mathbb{P}(A) = 0$.

2. We mimic the proof of Theorem 8.16. The main lines are as follws: Fix $C < \infty$. Let

$$A_n = \{\}\text{There is an } s \in [0,1] \text{ such that } |B_t - B_s| \le C|t-s|^{\frac{1}{2}+\frac{1}{k}} \text{ when } |t-s| \le \frac{k+1}{n}\}.$$

For $1 \le i \le n-k$, let $Y_{i,n} = \max\{|B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}}| : j = 0, \dots, k\}$. We try to estimate $Y_{i,n}$ on A_n. The worst case is when s is the right endpoint of an interval $(j/n, (j+1)/n)$. For exmaple, if $s = 1$, the worst case is

$$|B_{(n-k)/n} - B_{(n-k-1)/n}| \le |B_{(n-k)/n-B_1}| + |B_1 - B_{(n-k-1)/n}| \le C\left(\left(\frac{k}{n}\right)^{\frac{1}{2}+\frac{1}{k}} + \left(\frac{k+1}{n}\right)^{\frac{1}{2}+\frac{1}{k}}\right).$$

Let $B_n = \left\{\text{at least one } Y_{i,n} \le C\left(\left(\frac{k}{n}\right)^{\frac{1}{2}+\frac{1}{k}} + \left(\frac{k+1}{n}\right)^{\frac{1}{2}+\frac{1}{k}}\right)\right\}$. Then the above estimate shows $A_n \subseteq B_n$. And so

$$\mathbb{P}(A_n) \le \mathbb{P}(B_n) \le n\mathbb{P}\left[\frac{1}{\sqrt{n}}|B_1| \le C\left(\left(\frac{k}{n}\right)^{\frac{1}{2}+\frac{1}{k}} + \left(\frac{k+1}{n}\right)^{\frac{1}{2}+\frac{1}{k}}\right)\right]^{k+1} \le \text{constant} \cdot n^{-k} \to 0$$

as $n \to \infty$.

3. Part (a) is a direct computation

$$\mathbb{E}[\sum_{m=1}^{2^n} \Delta_{m,n}^2] = \sum_{m=1}^{2^n} \mathbb{E}[\Delta_{m,n}^2] = \sum_{m=1}^{2^n} t2^{-n} = t.$$

For part (b), we write $X_n = \sum_{m=1}^{2^n} \Delta_{m,n}$ for simplicity. The variance of X_n is

$$\mathbb{E}[(X_n - t)^2] = \sum_{m=1}^{2^n} \mathbb{E}[(\Delta_{m,n}^2 - t2^{-n})^2]$$

by independent increments of Brownian motion. And the variance is $2t^2 2^{-n}$. By Chebychev's inequality,

$$\mathbb{P}(|X_n - t| \geq \varepsilon) \leq \frac{2t^2}{\varepsilon^2} 2^{-n}$$

so

$$\sum \mathbb{P}(|X_n - t| \geq \varepsilon) \leq \infty.$$

Since

$$\mathbb{P}(X_n \to t) = 1 - \mathbb{P}\left(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{n} |X_i - t| > 1/m\right) \geq 1 - \sum_m \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{i=1}^{n} |X_i - t| > 1/m\right) = 1$$

by Borel-Cantelli, applied to $\varepsilon = 1/m$.

4. Part a comes from the fact that if X is $N(0, \sigma^2)$, then aX is $N(0, a^2\sigma^2)$. To do part b, wlog first assume $s \leq t$. Then by independent increment of Brownian motion,

$$\mathbb{E}[X_s X_t] = \mathbb{E}[e^{-\lambda(s+t)} B_{e^{2\lambda s}} B_{e^{2\lambda s}}] = e^{-\lambda(t-s)}$$

so we have the $\mathbb{E}[X_s X_t] = e^{-\lambda|t-s|}$. For Part c, observe that X_t and Y_t are Gaussian processes. Joint distribution of Guassians is determined by variances and covariances. So the FDD depends on $\mathbb{E}[Y_t]$ and $\text{cov}(Y_s Y_t)$, which are equal to $\mathbb{E}[X_t] = 0$ and $\text{cov}(X_s X_t)$ by part b.