280C HW2 Idea

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1. We try to compute the conditional expectation $\mathbb{E}[f(X_t)|\mathscr{F}_s]$. Given any bounded continuous function f,

$$\mathbb{E}[f(X_t)|\mathscr{F}_s] = \mathbb{E}[f(e^{-\lambda t}(B_{e^{2\lambda t}} - B_{e^{2\lambda s}})) + e^{-\lambda t}B_{e^{e^{2\lambda s}}}|\mathscr{F}_s].$$

   By independent increments of Brownian motion, we have the above equation

$$= \frac{1}{\sqrt{2\pi(e^{2\lambda t} - e^{2\lambda s})}} \int f(e^{-\lambda t}u + e^{-\lambda(t-s)}x) e^{-\frac{x^2}{2(e^{2\lambda t} - e^{2\lambda s})}} du \bigg|_{x=X_s}.$$

   After we do a substitution $y = e^{-\lambda t}u + e^{-\lambda(t-s)}x$, we get

$$\mathbb{E}[f(X_t)|\mathscr{F}_s] = \int f(y) \frac{1}{\sqrt{2\pi(1-e^{-2\lambda(s-t)})}} \exp\left(-\frac{(y - e^{-\lambda(s-t)}x)^2}{2(1-e^{-2\lambda(s-t)})}\right) dy \bigg|_{x=X_s}.$$

   This shows X_t is a Markov process and computed the Markov transition kernel

$$Q_{s,t}f(x) = \int f(y) \frac{1}{\sqrt{2\pi(1-e^{-2\lambda(s-t)})}} \exp\left(-\frac{(y - e^{-\lambda(s-t)}x)^2}{2(1-e^{-2\lambda(s-t)})}\right) dy$$

   or the $p(x, B)$ is given by

$$p(x, B) = \int_B \frac{1}{\sqrt{2\pi(1-e^{-2\lambda(s-t)})}} \exp\left(-\frac{(y - e^{-\lambda(s-t)}x)^2}{2(1-e^{-2\lambda(s-t)})}\right) dy$$

   Observe that it is indeed time homogeneuous.

2. Since if L is the largest zero of Brownian motion in $[0, 1]$ and $L \leq t$, then it takes time longer than $1-t$ to reach another zero from L, in terms of the shift map θ_t, we have

$$\mathbb{1}_{T_0 > 1-t} \circ \theta_t = \mathbb{1}_{L \leq t}.$$

   By Markov property,

$$\mathbb{E}_0[\mathbb{1}_{L \leq t}|\mathscr{F}_t^+] = \mathbb{E}_{B_t}[\mathbb{1}_{T_0 > 1-t}].$$

   Taking expectation on both sides, we have

$$\mathbb{P}(L \leq t) = \int \mathbb{E}_y[\mathbb{1}_{T_0 > 1-t}]\, p(t, 0, dy) = \int \mathbb{P}_y(T_0 > 1-t)\, p(t, 0, dy).$$

   The rest of the result follows from Example 8.4.2, page 374 of Durrett.

3. Since L is isolated from the right and almost surely zeros of B_t cannot be isloated, there must be zeros $z_n \uparrow L$. (Warning: almost surely there are zeros between z_n, otherwise the z_n is also isolated zero.) Since almost surely B_t cannot be constantly 0 in $[z_n, z_{n+1}]$, there must be $t_k \in [z_{n_k, z_{n_k+1}}]$ and $s_m \in [z_{n_m, z_{n_m+1}}]$ such that $B_(t_k) < 0$ and $B_{s_m} > 0$.

4. To show that it is a martingale,

$$\mathbb{E}[e^{i\lambda B_t}|\mathscr{F}_s] = e^{i\lambda B_s}\mathbb{E}[e^{i\lambda(B_t - B_s)}|\mathscr{F}_s].$$

   However, how the conditional expectation becomes expectation, by independent increments of Brownian motions.There are independent X, Y which are $N(0, t-s)$ such that $\mathbb{E}[e^{i\lambda(B_t - B - s)}] = \mathbb{E}[e^{i\lambda X}]\mathbb{E}[e^{-\lambda Y}]$ which is a product of characteristic function and moment generating function of $N(0, t-s)$. It follows that

$$\mathbb{E}[e^{i\lambda B_t}|\mathscr{F}_s] = e^{i\lambda B_s}\exp(-(t-s)\lambda^2/2)\exp((t-s)\lambda^2/2) = e^{i\lambda B_s}.$$

   If we assume $B_0 = i$, since $e^{i\lambda B_t}$ is a martingale,

$$\mathbb{E}[e^{i\lambda B_t}] = \mathbb{E}[e^{i\lambda B_0}] = e^{-\lambda}.$$

5. For $\mathbb{E}_x[\tau]$, the main observation is that

$$\mathbb{E}_x[T_R|T_R < T_0] = \mathbb{E}_x[T|T_R < T_0]$$

   where $T = T_R \wedge T_0$ since $B_t^2 - t$ is a martingale (and other technical assumptions are satisfied), the optional stopping theorem yields

$$x^2 = \mathbb{E}_x[B(\tau)^2 - \tau] = R^2 - \mathbb{E}_x[\tau]$$

   so $\mathbb{E}_x[\tau] = R^2 - x^2$. For $\mathbb{E}_x[T_R|T_R < T_0]$, first, we can check that $B_t^3 - 3tB_t$ is a martingale. Since $P_x(T_R < T_0) = x/R$ by theorem 8.5.3 Durrett, the optional stopping theorem (again technical assumptions are satisfied) yields

$$x^3 = \mathbb{E}_x[B(T)^3 - 3TB(T)] = x(R^2 - 3\mathbb{E}_x[T_R|T_R < T_0]).$$

   Hence,

$$\mathbb{E}_x[T_R|T_R < T_0] = \frac{1}{3}(R^2 - x^2)$$