Homework 1 Math280A Fall 2017

Due Friday in class, Oct 6. Relevant sections in Durrett's textbook: 1.1; in Resnick book: 1.6,1.7, 1.8, 2.1, 2,2. Justify all your answers.

1. (exercise 16 Resnick p22). Suppose $\mathcal{A}$ is a collection of subsets of Ω such that

- $\Omega \in \mathcal{A}$,
- $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$,
- if $A_1, ...A_n \in \mathcal{A}$ are *disjoint* subsets, then $\cup_{i=1}^n A_i \in \mathcal{A}$.

Show that $\mathcal{A}$ is not necessarily a field. (Hint: try a collection of two point subsets of $\Omega = \{1, 2, 3, 4\}$.)

2. Let Ω be a non-empty set and $\mathcal{C}$ be all one point subsets.

(i) Show that

$$\sigma(\mathcal{C}) = \{A \subseteq \Omega : \ A \text{ or } A^c \text{ is countable}\}.$$

(ii) Suppose in addition Ω is uncountable. Let $P : \sigma(\mathcal{C}) \to [0, 1]$ be defined as $P(A) = 0$ if A is countable; $P(A) = 1$ if A is not countable. Show that $(\Omega, \sigma(\mathcal{C}), P)$ is a probability space.

3. Exercise 1.1.4, Durrett page 9.

4. Exercise 1.1.5, Durrett page 9.

5. (from Exercise 1.1.6 Durrett) A subset A of $\mathbb{N} = \{1, 2, ...\}$ is said to have asymptotic density θ if the limit

$$\lim_{n\to\infty} |A \cap \{1, 2, ..., n\}|/n$$

exists and is equal to θ. Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Answer yes or no to the following:

- is $\mathcal{A}$ closed under taking complements?
- is $\mathcal{A}$ closed under taking finite union?
- is $\mathcal{A}$ closed under taking disjoint finite union?
- is $\mathcal{A}$ closed under taking disjoint countable union?

6. Let P be a probability measure on the Borel σ-field $\mathcal{B}(\mathbb{R})$ of the real line $\mathbb{R}$. Use the $\pi - \lambda$ theorem to show that for any $B \in \mathcal{B}(\mathbb{R})$, for any $\epsilon > 0$, there exists a set A which is a *finite* union of intervals such that

$$P(A \triangle B) < \epsilon.$$

Here $A\triangle B$ is the symmetric difference: $A\triangle B = (A \cap B^c) \cup (A^c \cap B)$.