Homework 2 Math280A Fall 2017

Due Friday in class, Oct 13. Relevant sections in Durrett's textbook: 1.1,1,2,1.3, appendix A1; in Resnick book: Chapter 2. Justify all your answers.

1. Suppose $\mathcal{C}$ is a class of subsets of Ω and suppose $B \subseteq \Omega$ satisfies that $B \in \sigma(\mathcal{C})$. Show that there exists a countable subclass $\mathcal{C}_B \subset \mathcal{C}$ such that $B \in \sigma(\mathcal{C}_B)$.

(Hint: This exercise is similar to Problem 6 on HW1. Set $\mathcal{L}$ to be the collection of subsets of Ω which satisfy the property described, use $\pi - \lambda$ theorem to show $\mathcal{L}$ contains $\sigma(\mathcal{C})$. Mind when you set up the proof that $\mathcal{C}$ is not necessarily a π-system.)

2. In a probability space $(\Omega, \mathcal{B}, P)$, call two sets A_1, A_2 equivalent if $P(A_1 \triangle A_2) = 0$. An atom in $(\Omega, \mathcal{B}, P)$ is defined as (the equivalence class of) a set $A \in \mathcal{B}$ such that $P(A) > 0$ and if $B \subseteq A$ and $B \in \mathcal{B}$, then either $P(B) = 0$ or $P(A \setminus B) = 0$.

i) If $\Omega = \mathbb{R}$ and P is determined by a distribution function $F(x)$, show that the atoms are the discontinuity points of F, that is $\{x : F(x) - F(x-) > 0\}$, where $F(x-)$ is the left limit of F at x.

ii) Let A, B be two atoms in $(\Omega, \mathcal{B}, P)$. Show that either $P(A \triangle B) = 0$ or $P(A \cap B) = 0$.

iii) Show that a probability space contains at most countably many atoms. In particular, a distribution function F has at most countably many discontinuities.

3. On $\mathbb{N} = \{1, 2, \ldots\}$, define $F = \{A \subset \mathbb{N} : A \text{ is finite or } A \text{ contains } \{n, n+1, ...\} \text{ for some } n \in \mathbb{N}\}$. Let $\mu : F \to [0, 1]$ be

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ 1 & \text{otherwise.} \end{cases}$$

i) Show that F is a field and μ is a finitely additive meausure on F (meaning for a finite disjoint collection $A_1, A_2, ...A_n \in F$, $\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$).

ii) Such a finitely additive measure of total mass 1 is often called a mean, it behaves differently from a probability measure. If $A_n \in F$ is a countable sequence of subsets such that $A_n \downarrow \emptyset$, is it true that $\mu(A_n) \downarrow 0$?

4. Let P be a probability measure on a field F_0. Recall in the Carathéodary extension theorem, we defined the outer measure P^* by

$$P^* : 2^\Omega \to [0, 1]$$

$$P^*(A) = \inf\{\sum_{n=1}^{\infty} P(B_n) : \ A \subseteq \cup_{n=1}^{\infty} B_n, \ B_n \in F_0 \text{ for all } n\}.$$

Also recall the σ-field $\mathcal{F}$ which consists of P^*-measurable sets.

i) Denote by P the probability measure on $(\Omega, \mathcal{F})$ which extends P on F_0. Show that $P^* : 2^\Omega \to [0,1]$ can be written as

$$P^*(A) = \inf\{P(B) : \; A \subseteq B, B \in \mathcal{F}\}.$$

Moreover, the infimum is always achieved.

ii) Similarly, one can define an inner measure

$$P_* : 2^\Omega \to [0,1]$$
$$P_*(A) = \sup\{P(B) : \; B \subseteq A, B \in \mathcal{F}\}.$$

Show that A is P^*-measurable (that is $A \in \mathcal{F}$) if and only if $P^*(A) = P_*(A)$.

5. (Exercise 1.3.5 on P16 Durrett) A function $f : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous (l.s.c.) if

$$\liminf_{y \to x} f(y) \geq f(x).$$

Show that f is l.s.c. if and only if $\{x : f(x) \leq a\}$ is closed for any $a \in \mathbb{R}$. Using this characterization, show that a l.s.c. function is measurable with respect to the Borel σ-field on $\mathbb{R}$.