Homework 3 Math280A Fall 2017

Due Friday in class, Oct 20. Relevant sections in Durrett's textbook: 1.3, 1.4, 1.5 ; in Resnick book: 3.2, 3.3,5.1, 5.2, 5.3. Justify all your answers.

1. (Exercise 1.3.7 Durrett P.16) Recall that a function $\varphi : \Omega \to \mathbb{R}$ is a simple function if φ can be written as

$$\varphi = \sum_{k=1}^{n} c_k \mathbf{1}_{A_k}$$

where $c_k \in \mathbb{R}$ and $A_k \in \mathcal{F}$. Show that the class of measurable functions $(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is the smallest class containing the simple functions and closed under taking pointwise limits.

(Hint: in this exercise need to show (i) the class of $\mathcal{F}$-measurable functions is closed under taking pointwise limits (ii) every $\mathcal{F}$-measurable function can be realized as pointwise limit of a sequence of simple functions.)

2. Let $X : \Omega \to \mathbb{R}$ be a random variable on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the σ-field generated by X as

$$\sigma(X) = \sigma\left(\left\{X^{-1}(B) : \ B \in \mathcal{B}(\mathbb{R})\right\}\right).$$

(i) Show that $\sigma(X)$ is a countably generated σ-field.

(ii) Show that a function $Y : (\Omega, \sigma(X)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable if and only if there exists a measurable function $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $Y = f(X)$. (Hint: you can apply the result of the previous exercise)

3. (Exercise 1.4.1. Durrett) Let $X : \Omega \to \mathbb{R}$ be a random variable on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $X \geq 0$ and $\mathbb{E}[X] = 0$, then $X = 0$ a.s.: $\mathbb{P}(X = 0) = 1$.

4. (application of monotone convergence theorem)

Let $f : \Omega \to \mathbb{R}$ be a measurable function on $(\Omega, \mathcal{F}, \mu)$. Suppose $f \geq 0$. Define $\nu_f : \mathcal{F} \to \mathbb{R}$ by

$$\nu(A) = \int_A f d\mu = \int f \mathbf{1}_A d\mu.$$

(i) Show that ν is a measure on $(\Omega, \mathcal{F})$.

(ii) Suppose $\int f d\mu < \infty$. Show that $\forall \epsilon > 0$, there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $v(A) < \epsilon$. (When this property is satisfied, we say ν is absolutely continuous w.r.t. μ).

5. (Exercise 1.5.10 Durrett P27) Use dominated convergence to show that if $\sum_n \int |f_n| d\mu < \infty$ then

$$\sum_n \int f_n d\mu = \int \left(\sum_n f_n\right) d\mu.$$