280C HW3 Idea

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1. Write $X_t = (1+t)^{-1/2} e^{\frac{B_t^2}{2(1+t)}}$. To show that it is a martingale, we consider the conditional expectation

$$\mathbb{E}[(1+t)^{-1/2} e^{\frac{B_t^2}{2(1+t)}} | \mathscr{F}_s] = \mathbb{E}[(1+t)^{1/2} e^{\frac{((B_t-B_s)+B_s)^2}{2(1+t)}} | \mathscr{F}_s].$$

Since $B_t - B_s$ is indepdent of B_s, we have

$$= \int (1+t)^{-1/2} e^{\frac{(x+B_s)^2}{2(1+t)}} \frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{-x^2}{2(t-s)}}\, dx = e^{\frac{B_s^2}{2(s+1)}} \int (1+t)^{-1/2} e^{\frac{-(s+1)(x-\frac{t-s}{s+1}B_s)^2}{2(1+t)(t-s)}}\, dx$$

which is $(1+s)^{1/2} e^{\frac{B_s^2}{2(1+s)}}$. This shows it is a martingale.

By martingale convergence theorem, $X_t \to X_\infty$ a.s. By Durrett theorem 8.2.7, X_∞ is a constant a.s.

Let $Y_t = \frac{B_t}{((1+t)\log(1+t)))^{1/2}}$.Then

$$X_t = (1+t)^{-1/2} \exp(Y_t^2 \log(1+t)/2).$$

If $c := \limsup Y_t > 1$ a.s., we pick any $1 < c_0 < c$, then there are infinite many $t's$ such that $X_t > (1+t)^{(c_0^2-1)/2}$. This implies that it is impossible to have X_t convergent a.s. So, we must have $c \leq 1$.

2. Write $Z_n = m(\{t \in [0,1] : S_n^*(t) > 0\}) - \frac{1}{n}\#\{k \leq n : S_k > 0\}$. Given any $\delta > 0$, we look at the proof of Durrett example 8.6.4 with $a = 0$: on $\{\max_{m\leq n} |X_m| \leq \delta\sqrt{n}\}$,

$$m\{t \in [0,1] : S_n^*(t) > \delta\} \leq \frac{1}{n}\#\{k \leq n : S_k > 0\} \leq m\{t \in [0,1] : S_n^*(t) > -\delta\}.$$

So, on $\{\max_{m\leq n} |X_m| \leq \delta\sqrt{n}\}$,

$$\begin{aligned} &m\{t \in [0,1] : S_n^*(t) > \delta\} - m\{t \in [0,1] : S_n^*(t) > 0\} \\ \leq &- Z_n \leq m\{t \in [0,1] : S_n^*(t) > -\delta\} - m\{t \in [0,1] : S_n^*(t) > 0\}. \end{aligned}$$

Because (we denote m as Lebesgue measure on $[0,1]$ for convenience)

$$m\{S_n^*(t) > \delta\} - m\{S_n^*(t) > 0\} \to m\{B_t > \delta\} - m\{B_t > 0\}$$

and

$$m\{S_n^*(t) > -\delta\} - m\{S_n^*(t) > 0\} \to m\{B_t > -\delta\} - m\{B_t > 0\},$$

by continuity of $a \mapsto m\{B_t > a\}$ (see Durrett example 8.6.4), we can choose δ small enough such that $|Z_n| < \varepsilon$ for n large enough.

Since

$$\mathbb{P}(|Z_n| > \varepsilon) = \mathbb{P}(|Z_n| > \varepsilon, \max_{m\le n} |X_m| \le \delta\sqrt{n}) + \mathbb{P}(|Z_n| > \varepsilon, \max_{m\le n} |X_m| > \delta\sqrt{n})$$

where the latter term goes to 0 using Chebyshev's inequality, we have, by bounded convergence theorem, for any $\varepsilon > 0$,

$$\begin{aligned}
&\limsup \mathbb{P}(|Z_n| > \varepsilon) \\
\le &\limsup \mathbb{P}(|Z_n| > \varepsilon, \max_{m\le n} |X_m| \le \delta\sqrt{n}) + \limsup \mathbb{P}(|Z_n| > \varepsilon, \max_{m\le n} |X_m| > \delta\sqrt{n}) \\
= &0.
\end{aligned}$$

This proves the convergence in probability.

3. Let $Q_n = m(\{t \in [0,1] : S_n^*(t) > 0\})$. By Donsker's theorem $Q_n \Rightarrow m(\{t \in [0,1] : B(t) > 0\}) =: R$ and R is arcsine distributed. We decompose, for any $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{P}(P_n/n \le x) &= \mathbb{P}((P_n/n - Q_n) + P_n/n \le x) \\
&= \mathbb{P}((P_n/n - Q_n) > \varepsilon, P_n/n \le x) + \mathbb{P}((P_n/n - Q_n) \le \varepsilon, P_n/n \le x).
\end{aligned}$$

The first term goes to 0 by problem 2. For the second term, we know that $\mathbb{P}(P_n/n - Q_n \le \varepsilon) \to 1$ by problem 1. We also have

$$\{Q_n \le x - \varepsilon\} \le \{|P_n/n - Q_n| \le \varepsilon, P_n/n \le x\} \subseteq \{Q_n \le x + \varepsilon\}.$$

So, taking limit as $n \to \infty$, we have

$$\mathbb{P}(R \le x - \varepsilon) \le \lim_{n\to\infty} \mathbb{P}(P_n/n \le x) \le \mathbb{P}(R \le x + \varepsilon).$$

To conclude our proof, we let $\varepsilon \to 0$ and use the fact that the distribution function of arcsine law is *continuous* at *every* x (not simply right-continuous).

4. $H(x) > 0$ for all $x \ne a$ since the Markov chain is irreducible. Moreover, the function H is harmonic by Levin-Peres-Wilmer proposition 9.1, taking $h_{a,b}(x) = \mathbb{1}_{x=b}$ in proposition 9.1.

   We denote $\tau_b = \min n : X_n = b$. The $\hat{P}$ for $\{Y_j\}$ has the same law as $\{X_n\}$ conditioned on reaching b before a and being absorbed at b because for any $x \ne b$,

$$\hat{P}(x,y) = \frac{P(x,y)\mathbb{P}_y(T_{a,b} = \tau_b)}{\mathbb{P}_x(T_{a,b} = \tau_b)} = \frac{\mathbb{P}_x(X_1 = y, T_{a,b} = \tau_b)}{\mathbb{P}_x(T_{a,b} = \tau_b)} = \mathbb{P}_x(X_1 = y | T_{a,b} = \tau_b).$$

   Note that x cannot be taken as a in the above equations (since $\tau_b = T_{a,b}$) and we are not computing $x = b$ case.