1. A useful lower bound (second moment method): let \( Y \geq 0 \) be a non-negative random variable with \( \mathbb{E}Y^2 < \infty \). Show that
\[
\mathbb{P}(Y > 0) \geq \frac{(\mathbb{E}Y)^2}{\mathbb{E}Y^2}.
\]

2. (Exercise 1.6.8 Durrett Page 35) Suppose that the probability measure \( \mu \) has a density \( f \), that is
\[
\mu(A) = \int_A f(x)dx \quad \text{for any Borel set } A.
\]
Show that for any \( g \geq 0 \) or \( g \) with
\[
\int |g(x)| \mu(dx) < \infty,
\]
we have
\[
\int g(x) \mu(dx) = \int g(x)f(x)dx.
\]

3. Give an example of a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and two collections of events \(A_1\) and \(A_2\) such that \(A_1\) and \(A_2\) are independent, but the the \(\sigma\)-fields \(\sigma(A_1)\) and \(\sigma(A_2)\) are not independent. (Hint: consider a finite sample space \(\Omega\))

4. (Dyadic expansion of a random number) Let \(([0, 1], \mathcal{B}, \mathbb{P})\) be the probability space on \([0, 1]\) where \(\mathbb{P}\) is the Lebesgue (uniform) measure. Define
\[
Y_n : \Omega \to \{0, 1\}
\]
\[
Y_n(\omega) = \begin{cases} 
1 & \text{if } \lfloor 2^n \omega \rfloor \text{ is odd} \\
0 & \text{if } \lfloor 2^n \omega \rfloor \text{ is even.}
\end{cases}
\]
Here \(\lfloor x \rfloor\) is the largest integer \(n\) such that \(n \leq x\). Show that \(Y_1, Y_2, \ldots\) are independent with \(\mathbb{P}(Y_n = 0) = \mathbb{P}(Y_n = 1) = 1/2\).

5. Let \(([0, 1], \mathcal{B}, \mathbb{P})\) be the probability space on \([0, 1]\) where \(\mathbb{P}\) is the Lebesgue measure. Let \(X : \Omega \to \mathbb{R}\) be the uniform random variable \(X(\omega) = \omega\).
   (i) Does there exists a bounded random variable \(Y : \Omega \to \mathbb{R}\) such that \(Y\) is independent of \(X\) and \(Y\) is not constant \(\mathbb{P}\)-almost everywhere?
   (ii) Let \(Z = (X - 1/2)^2\). Construct a random variable \(Y : \Omega \to \mathbb{R}\) such that \(Y\) is independent of \(Z\) and \(Y\) is not constant \(\mathbb{P}\)-a.e.