Homework 4 Math280A Fall 2017

Due Friday in class, Oct 27. Relevant sections in Durrett's textbook 1.6, 2.1; in Resnick book: 4.1, 4.2. Justify all your answers.

1. A useful lower bound (second moment method): let $Y \geq 0$ be a non-negative random variable with $\mathbb{E}Y^2 < \infty$. Show that

$$\mathbb{P}(Y > 0) \geq \frac{(\mathbb{E}Y)^2}{\mathbb{E}Y^2}.$$

2. (Exercise 1.6.8 Durrett Page 35) Suppose that the probability measure μ has a density f, that is $\mu(A) = \int_A f(x)dx$ for any Borel set A. Show that for any $g \geq 0$ or g with $\int |g(x)|\,\mu(dx) < \infty$, we have

$$\int g(x)\mu(dx) = \int g(x)f(x)dx.$$

3. Give an example of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two collections of events $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent, but the the σ-fields $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are not independent. (Hint: consider a finite sample space Ω)

4. (Dyadic expansion of a random number) Let $([0,1], \mathcal{B}, \mathbb{P})$ be the probability space on $[0,1]$ where $\mathbb{P}$ is the Lebesgue (uniform) measure. Define

$$Y_n : \Omega \to \{0,1\}$$
$$Y_n(\omega) = \begin{cases} 1 & \text{if } \lfloor 2^n\omega \rfloor \text{ is odd} \\ 0 & \text{if } \lfloor 2^n\omega \rfloor \text{ is even.} \end{cases}$$

Here $\lfloor x \rfloor$ is the largest integer n such that $n \leq x$. Show that $Y_1, Y_2, \ldots$ are independent with $\mathbb{P}(Y_n = 0) = \mathbb{P}(Y_n = 1) = 1/2$.

5. Let $([0,1], \mathcal{B}, \mathbb{P})$ be the probability space on $[0,1]$ where $\mathbb{P}$ is the Lebesgue measure. Let $X : \Omega \to \mathbb{R}$ be the uniform random variable $X(\omega) = \omega$.

(i) Does there exists a bounded random variable $Y : \Omega \to \mathbb{R}$ such that Y is independent of X and Y is not constant $\mathbb{P}$-almost everywhere?

(ii) Let $Z = (X - 1/2)^2$. Construct a random variable $Y : \Omega \to \mathbb{R}$ such that Y is independent of Z and Y is not constant $\mathbb{P}$-a.e.