Homework 5 Math280A Fall 2017

Due Friday in class, Nov 3. Relevant sections in Durrett's textbook 1.7, 2.1, 2.3; in Resnick book: 5.7, 5.8, 5.9, 4.5.1. Justify all your answers.

1. Use Fubini's theorem to show that for a distribution function $F : \mathbb{R} \to [0,1]$,

$$\int_{\mathbb{R}} (F(x+c) - F(x))dx = c$$

for $c \in \mathbb{R}$. Here "dx" denotes integration with respect to Lebesgue measure.

2. Show that the product σ-field is the smallest σ-field making the coordinate mappings $\pi_1 : \Omega_1 \times \Omega_2 \to \Omega_1$ and $\pi_2 : \Omega_1 \times \Omega_2 \to \Omega_2$ measurable.

3. Let μ and ν be two probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let F and G be their corresponding distribution functions, $F(x) = \mu((-\infty, x])$ and $G(x) = \nu((-\infty, x])$.

(i) Use Fubini's theorem to show that

$$\int_{(a,b]} (F(y) - F(a))dG(y) = (\mu \times \nu)\left(\{(x,y) : \; a < x \le y \le b\}\right).$$

(ii) Deduce from (i) (be careful with discontinuities of $F.G$) that

$$\int_{(a,b]} F(y)dG(y) + \int_{(a,b]} G(y)dF(y)$$
$$= F(b)G(b) - F(a)G(a) + \sum_{x \in (a,b]} \mu(\{x\})\nu(\{x\}).$$

(iii) Show that if X_1 and X_2 are independent with common distribution function F, where F is continuous, then

$$\mathbb{P}(X_1 \le X_2) = \frac{1}{2}.$$

4. Suppose $\{A_n\}$ are independent events satisfying $P(A_n) < 1$ for all n. Show that

$$P\left(\cup_{n=1}^{\infty} A_n\right) = 1 \text{ if and only if } P(A_n \; i.o.) = 1.$$

Give an example to show that the condition $P(A_n) < 1$ cannot be dropped.

5. Use the Borel-Cantelli Lemma to show that given any sequence of random variables $\{X_n, n \geq 1\}$ whose range is the whole real line (meaning for any $c > 0$, $P(X_n > c) > 0$ and $P(X_n < -c) > 0$), there exists constants $c_n \to \infty$ such that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{X_n}{c_n} = 0\right) = 1.$$

Give a detailed description of how you choose the sequence c_n.