Due Monday in class, Nov 13. Relevant sections in Durrett’s textbook 2.3, 5.5; in Resnick book: Chapter 6. Justify all your answers.

1. (Fatou’s lemma and dominated convergence theorem for convergence in probability)
   (i) Suppose $X_n \geq 0$ and $X_n \to X$ in probability. Then $\liminf_{n \to \infty} E X_n \geq E X$.
   (ii) Suppose $X_n \to X$ in probability and $|X_n| \leq Y$ with $E Y < \infty$. Then $\lim_{n \to \infty} E X_n = E X$.

2. This exercise shows convergence in probability is metrizable. On the space of random variables define a metric
   \[ d(X,Y) = E \left[ \frac{|X - Y|}{1 + |X - Y|} \right]. \]
   By definition $d$ is obviously symmetric, $d(X,Y) = d(Y,X)$. By a previous homework exercise, $d(X,Y) = 0$ if and only if $\frac{|X - Y|}{1 + |X - Y|} = 0$ a.s., that is a.s. $X = Y$.
   (i) To verify $d$ is a metric, show the triangle inequality $d(X,Z) \leq d(X,Y) + d(Y,Z)$.
   (ii) Show that if $X_n \to 0$ in probability if and only if $E \left[ \frac{|X_n|}{1 + |X_n|} \right] \to 0$. It follows that $X_n \to X$ in probability if and only if $d(X_n,X) \to 0$ as $n \to \infty$.

3. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, P)$ where $\Omega$ is countable and $\mathcal{F}$ is the $\sigma$-field that consists of all subsets of $\Omega$. Show that on this countable probability space almost sure convergence and convergence in probability are equivalent: $X_n \to X$ in probability implies $X_n \to X$ a.s.

4. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with distribution function $F$. Let $(\lambda_n)$ be an increasing sequence of numbers and define
   \[ A_n = \{ \max_{1 \leq m \leq n} X_m > \lambda_n \}. \]
   Show that $P(A_n i.o.) = 0$ if $\sum_{n=1}^{\infty} (1 - F(\lambda_n)) < \infty$; and $P(A_n i.o.) = 1$ if $\sum_{n=1}^{\infty} (1 - F(\lambda_n)) = \infty$. 

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5. Suppose $X_1, X_2, \ldots$ is an uncorrelated sequence, that is $Cov(X_i, X_j) = 0$ for any $i \neq j$. Further assume that $E X_i = 0$, $Var(X_i) = C \in (0, \infty)$ for all $i \in \mathbb{N}$, that is all random variables have mean 0 and the same finite non-zero variance.

Let $(a_n)$ be a sequence of real numbers. Show that $S_n = \sum_{i=1}^{n} a_i X_i$ is convergent in $L^2$ if and only if $\sum_{i=1}^{\infty} a_i^2 < \infty$.

6. Suppose $\{X_n\}$ and $\{Y_n\}$ are two families of uniform integrable variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Is the family $\{X_n + Y_n\}$ uniformly integrable?