Homework 6 Math280A Fall 2017

Due *Monday* in class, Nov 13. Relevant sections in Durrett's textbook 2.3, 5.5; in Resnick book: Chapter 6. Justify all your answers.

1. (Fatou's lemma and dominated convergence theorem for convergence in probability)

(i) Suppose $X_n \geq 0$ and $X_n \to X$ in probability. Then $\liminf_{n\to\infty} \mathbb{E}X_n \geq \mathbb{E}X$.

(ii) Suppose $X_n \to X$ in probability and $|X_n| \leq Y$ with $\mathbb{E}Y < \infty$. Then $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.

2. This exercise shows convergence in probability is metrizable. On the space of random variables define a metric

$$d(X,Y) = \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right].$$

By definition d is obviously symmetric, $d(X,Y) = d(Y,X)$. By a previous homework exercise, $d(X,Y) = 0$ if and only if $\frac{|X-Y|}{1+|X-Y|} = 0$ a.s., that is a.s. $X = Y$.

(i) To verify d is a metric, show the triangle inequality

$$d(X,Z) \leq d(X,Y) + d(Y,Z).$$

(ii) Show that if $X_n \to 0$ in probability if and only if $\mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] \to 0$. It follows that $X_n \to X$ in probability if and only if $d(X_n, X) \to 0$ as $n \to \infty$.

3. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is *countable* and $\mathcal{F}$ is the σ-field that consists of all subsets of Ω. Show that on this countable probability space almost sure convergence and convergence in probability are equivalent: $X_n \to X$ in probability implies $X_n \to X$ a.s.

4. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with distribution function F. Let (λ_n) be an increasing sequence of numbers and define

$$A_n = \{\max_{1\leq m\leq n} X_m > \lambda_n\}.$$

Show that $\mathbb{P}(A_n \ i.o.) = 0$ if $\sum_{n=1}^{\infty}(1-F(\lambda_n)) < \infty$; and $\mathbb{P}(A_n \ i.o.) = 1$ if $\sum_{n=1}^{\infty}(1-F(\lambda_n)) = \infty$.

5. Suppose $X_1, X_2, \ldots$ is an uncorrelated sequence, that is $Cov(X_i, X_j) = 0$ for any $i \neq j$. Further assume that $\mathbb{E}X_i = 0$, $Var(X_i) = C \in (0, \infty)$ for all $i \in \mathbb{N}$, that is all random variables have mean 0 and the same finite non-zero variance.

Let (a_n) be a sequence of real numbers. Show that $S_n = \sum_{i=1}^{n} a_i X_i$ is convergent in L^2 if and only if $\sum_{i=1}^{\infty} a_i^2 < \infty$.

6. Suppose $\{X_n\}$ and $\{Y_n\}$ are two families of uniform integrable variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Is the family $\{X_n + Y_n\}$ uniformly integrable?