# Homework 7 Math280A Fall 2017

Due *Monday* in class, Nov 20. Relevant sections in Durrett's textbook 2.2, 2.3; in Resnick book: 6.5, Chapter 7. Justify all your answers.

1. Suppose $X_n$ is a sequence of non-negative random variables, that is $X_n \geq 0$ for all $n$. Suppose $X_n \to X$ in probability when $n \to \infty$, where $X$ is also a non-negative random variable. Further assume that $\mathbb{E}X < \infty$, $\mathbb{E}X_n < \infty$ and $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$. Show that $X_n \to X$ in $L^1$. (*Hint:* first show that the family $(X - X_n)_+$ is uniformly integrable.)

2. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = (-1)^k k) = \frac{c}{k^2 \log k}$ for $k \geq 2$, where $c$ is the normalization constant so that the sum of probabilities is 1. Let $S_n = X_1 + \ldots + X_n$. Show that $\mathbb{E}|X_i| = \infty$, but there is a finite constant $\mu$ so that $S_n/n \to \mu$ in probability.

3. Suppose $X_n$ is a sequence of i.i.d. random variables such that $0 \leq X_n < \infty$ and $\mathbb{P}(X_n > x) > 0$ for all $x > 0$. Let $M(s)$ be the truncated first moment:

$$M(s) = \mathbb{E}\left[X_n \mathbf{1}_{\{X_n \leq s\}}\right].$$

**(i)** Verify that $s/M(s) \to \infty$ when $s \to \infty$.

**(ii)** Set

$$b_n = \inf\{s : \ s/M(s) \geq n\}.$$

Show that if

$$n\mathbb{P}(X_1 > b_n) \to 0 \text{ when } n \to \infty,$$

then $S_n/b_n \to 1$ in probability. (*Hint:* use the weak law for triangular arrays.)

4. Let $X_0$ be the (deterministic) vector $X_0 = (1, 0) \in \mathbb{R}^2$. Define a sequence of random variables $X_n$ iteratively by taking $X_{n+1}$ to be chosen uniformly from the ball of radius $|X_n|$ centered at the origin, and $X_{n+1}/|X_n|$ is independent of $X_1, \ldots, X_n$. Show that $\frac{1}{n} \log |X_n|$ converges a.s. and find the limit.

5. Let $X_n$ be a sequence of i.i.d. random variables with $\mathbb{E}|X_n| < \infty$ and $\mathbb{E}X_n \neq 0$. Show that

$$\frac{\max_{1\leq i\leq n} |X_i|}{|S_n|} \to 0 \ a.s.$$