Homework 8 Math280A Fall 2017

Due *Wednesday* in class, Nov 29. Relevant sections in Durrett's textbook 2.4, 2.5; in Resnick book: Chapter 7. Justify all your answers.

1. (Betting on favorable game). Suppose you start with \$1. On each bet, independently win the amount of your bet with probability $\frac{1}{2}+q$ and lose with probability $\frac{1}{2}-q$, $q \in (0, \frac{1}{2})$. Assume we always bet proportion $a \in (0,1]$ of our current fortune. What is the optimal choice of a as a function of q?

2. Prove that the three series theorem reduces to the following when the random variables are positive. If $X_n \geq 0$ are independent, then $\sum_n X_n < \infty$ a.s. if and only if for any $c > 0$, we have

$$\sum_n \mathbb{P}(X_n > c) < \infty,$$

$$\sum_n \mathbb{E}\left(X_n \mathbf{1}_{\{X_n \leq c\}}\right) < \infty.$$

3. Suppose $\{X_n, n \geq 1\}$ are independent random variables with $\mathbb{E}[X_n] = 0$ for all n. If

$$\sum_n \mathbb{E}\left(X_n^2 \mathbf{1}_{\{|X_n| \leq 1\}} + |X_n| \mathbf{1}_{\{|X_n| > 1\}}\right) < \infty,$$

then $\sum_n X_n$ converges a.s.

4. Suppose $\{X_n, n \geq 1\}$ are independent with distributions

$$\mathbb{P}(X_n = n^{-\alpha}) = \mathbb{P}(X_n = -n^{-\alpha}) = \frac{1}{2}.$$

Use the Kolmogorov convergence criterion to verify that if $\alpha > 1/2$, then $\sum_n X_n$ converges a.s. Use the three series theorem to verify that $\alpha > 1/2$ is necessary for convergence.

5. Let $X_1, X_2, \ldots$ be i.i.d. and not constant 0. Let $r(\omega)$ be the radius of convergence of the power series $\sum_{n \geq 1} X_n(\omega) z^n$, that is

$$r(\omega) = \sup\{s \in \mathbb{R}_{\geq 0} : \sum |X_n(\omega)| s^n < \infty\}.$$

Show that $r(\omega) = 1$ a.s. or $r(\omega) = 0$ a.s., according to $\mathbb{E}(\log_+ |X_1|) < \infty$ or $= \infty$. Here $\log_+ x = \max\{0, \log x\}$.