Homework 9 Math280A Fall 2017

Due *Wednesday* in class, Dec 6. Relevant sections in Durrett's textbook 2.5 2.6; in Resnick book: Chapter 7. Justify all your answers.

1. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}|X_1| < \infty$. Suppose (c_n) is a bounded sequence of real numbers. Show that

$$\frac{1}{n}\sum_{i=1}^{n} c_j X_j \to 0 \text{ a.s.}$$

2. Let $X_1, X_2, \ldots$ be independent random variables. Let $S_n = X_1 + \ldots + X_n$, $S_{m,n} = S_n - S_m$ where $m \le n$.

(i) Show that

$$\mathbb{P}\left(|S_{m,n}| \ge a\right) \ge \mathbb{P}\left(\max_{m<j\le n} |S_{m,j}| \ge 2a\right) \cdot \min_{m<k\le n} \mathbb{P}\left(|S_{k,n}| \le a\right).$$

(ii) Use (i) to show the following: if $S_n \to W$ in probability, where W is the limiting random variable, then S_n also converges to W a.s. (*Hint: show that (S_n) is a Cauchy sequence a.s.*)

3. Let $X_1, X_2, \ldots$ be i.i.d. Poisson random variables with mean 1. Find

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(S_n \ge na\right)$$

for $a > 1$, where $S_n = X_1 + \ldots + X_n$.

4. Suppose X is a random variable such that

$$\varphi(t) = \mathbb{E}e^{tX} < \infty \text{ for all } t \in \mathbb{R}.$$

Use Hölder inequality and Fatou's lemma to show that $\log\varphi$ is convex and lower semi-continuous on $\mathbb{R}$.