MATH 180 B

Introduction to Stochastic Processes I

What do we cover in 180B

- Conditional probability distribution
  Multivariate normal distributions

- Markov chains
  - transition prob., hitting prob.
  - stationary distribution, long term behavior
  - branching processes

- Poisson processes

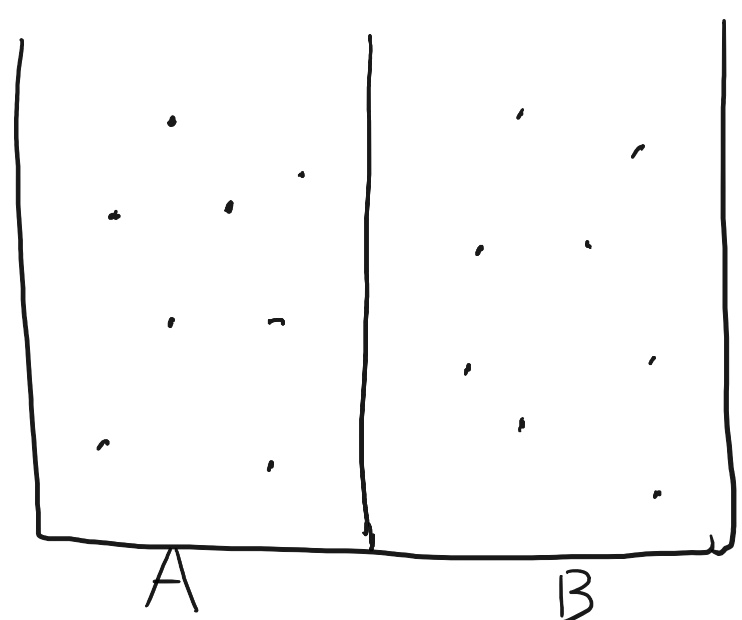
What is a stochastic process

A stochastic process is a collection of random variables $\{X_t\}_{t \in I}$

e.g. $I = 0, 1, 2, 3, \ldots$

$I = [0, \infty)$

Example: Ehrenfest Urn Model



We have $2a$ molecules in total

start with a molecules on left

a right

time is discrete $0, 1, 2, \ldots$

at each time j, pick up a molecule uniformly random from $2a$ molecules, and move it to the other side.

X_n = the number of molecules in A (left) side at time n.

Logistics

- Course website (for information of office hours, course schedule)

http://www.math.ucsd.edu/~tiz161/180b.html

- Course contents (notes, homework assignments&solutions, etc) will be posted on TritonEd.

- Weekly homework will count for 20 percent of the final grade. The lowest homework score will be dropped.
- Submit your homework on gradescope.
- There will be two in class midterm exams and a final exam.
- Each midterm will count for 20 percent, and the final exam will count for 40 percent; alternatively, you may drop one lower midterm and the final exam will count for 60 percent.

Recommended Textbooks

- *An Introduction to Stochastic Modeling* by Mark Pinsky and Samuel Karlin.

- *Essentials of Stochastic Processes* by Rick Durrett, which is available online through the UCSD library.

Conditional distributions (discrete case)

Recall:

- Two events A and B in sample space Ω

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

$\uparrow$

conditional probability

- X, Y two discrete random variables

$$X : \Omega \to \mathbb{R}, \quad Y : \Omega \to \mathbb{R}$$

X takes value in $\{x_1, x_2, \dots\}$

Y $\{y_1, y_2, \dots\}$

probability mass functions:

$$p_X : \{x_1, x_2, \dots\} \to [0,1]$$

$$p_X(x_i) = \mathbb{P}(X = x_i).$$

$$p_Y : \{y_1, y_2, \dots\} \to [0,1]$$

$$p_Y(y_i) = \mathbb{P}(Y = y_i).$$

The joint prob. mass function of (X, Y)

$$p_{X,Y} : \{(x_i, y_j)\} \to [0,1]$$

$$p_{X,Y}(x_i, y_j) = \mathbb{P}(X = x_i \text{ and } Y = y_j).$$

Example

X takes value in $\{0, 1, 2\}$

Y takes value in $\{10, 20, 30\}$

- Suppose X and Y have joint probability mass function:

$$f(0,10) = f(0,20) = \frac{2}{18},\ f(1,10) = f(1,30) = \frac{3}{18},\ f(1,20) = \frac{4}{18},\ f(2,30) = \frac{4}{18}.$$

	10	20	30
0	$\frac{2}{18}$	$\frac{2}{18}$	0
1	$\frac{3}{18}$	$\frac{4}{18}$	$\frac{3}{18}$
2	0	0	$\frac{4}{18}$

- The conditional probabilities are

Given $\{Y = 10\}$

What is the conditional distribution of X?

$$\mathbb{P}(X=1, Y=10) = \frac{3}{18}$$

$$\mathbb{P}(Y = 10) = \mathbb{P}(X=0, Y=10) + \mathbb{P}(X=1, Y=10) + \mathbb{P}(X=2, Y=10) = \frac{2}{18} + \frac{3}{18} = \frac{5}{18}$$

$$\mathbb{P}(X = 1 \mid Y = 10) = \frac{\mathbb{P}(X=1, Y=10)}{\mathbb{P}(Y=10)} = \frac{\frac{3}{18}}{\frac{5}{18}} = \frac{3}{5}.$$

$\mathbb{P}(X=0 \mid Y=10) = \frac{2}{5}$, $\mathbb{P}(X=2 \mid Y=10) = 0$.

Basic properties of conditional distribution

Def: let X, Y be two discrete R.V. on Ω

P_X p.m.f. of X, P_Y p.m.f. of Y

$P_{X,Y}$ the joint prob. mass function of (X, Y).

If $\mathbb{P}(Y=y) > 0$, then the conditional probability mass function of X given $\{Y=y\}$ is

$$P_{X|Y}(\cdot \mid y) : \{x_1, x_2, \dots\} \to [0,1]$$

$$P_{X|Y}(x_i \mid y) \stackrel{\text{def}}{=} \frac{\mathbb{P}(X=x_i, Y=y)}{\mathbb{P}(Y=y)}$$

$$= \mathbb{P}(X=x_i \mid Y=y).$$

For a pair of random variable X, Y

if we know the p.m.f. of Y

and know the conditional p.m.f. of X

given Y : $P_{X|Y}(\cdot|\cdot)$

Then we can recover the marginal distribution of X.

X takes value in $\{x_1, x_2, \dots\}$ Y $\{y_1, y_2, \dots\}$

$$P_X(x_i) = \mathbb{P}(X = x_i) = \sum_j \mathbb{P}(X = x_i, Y = y_j)$$

$$= \sum_{j: P(Y=y_j) \neq 0} \mathbb{P}(X = x_i | Y = y_j) \mathbb{P}(Y = y_j) = \sum_{j: P(Y=y_j) \neq 0} P_{X|Y}(x_i | y_j) P_Y(y_j).$$