MATH 180 B
Introduction to Stochastic Processes I
What do we cover in 180B

- Conditional probability distribution
  Multivariate normal distributions

- Markov chains
  - transition prob., hitting prob.
  - stationary distribution, long term behavior
  - branching processes

- Poisson processes
What is a stochastic process

A stochastic process is a collection of random variables \( \{ X_t \}_{t \in I} \)

e.g. \( I = 0, 1, 2, 3, \ldots \)

\( I = [0, \infty) \)

Example: Ehrenfest Urn Model

We have 2A molecules in total start with A molecules on left a
time is discrete $0, 1, 2, \ldots$

at each time $j$, pick up a molecule uniformly random from $2a$ molecules, and move it to the other side.

$X_n =$ the number of molecules in $A$ (left) side at time $n$. 
Logistics

- Course website (for information of office hours, course schedule)

http://www.math.ucsd.edu/~tiz161/180b.html

- Course contents (notes, homework assignments&solutions, etc) will be posted on TritonEd.
- Weekly homework will count for 20 percent of the final grade. The lowest homework score will be dropped.
- Submit your homework on gradescope.
- There will be two in class midterm exams and a final exam.
- Each midterm will count for 20 percent, and the final exam will count for 40 percent; alternatively, you may drop one lower midterm and the final exam will count for 60 percent.
Recommended Textbooks

- *An Introduction to Stochastic Modeling* by Mark Pinsky and Samuel Karlin.

- *Essentials of Stochastic Processes* by Rick Durrett, which is available online through the UCSD library.
Conditional distributions (discrete case)

Recall:

- Two events $A$ and $B$ in sample space $\Omega$

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)},
\]

conditional probability

- $X, Y$ two discrete random variables

\[
X : \Omega \rightarrow \mathbb{R}, \quad Y : \Omega \rightarrow \mathbb{R}
\]
X takes value in \{x_1, x_2, \ldots \}
Y \{y_1, y_2, \ldots \}

Probability mass functions:

\[ P_X : \{x_1, x_2, \ldots \} \rightarrow [0, 1] \]
\[ P_X(x_i) = P(X = x_i). \]

\[ P_Y : \{y_1, y_2, \ldots \} \rightarrow [0, 1] \]
\[ P_Y(y_i) = P(Y = y_i). \]

The joint prob. mass function of \((X, Y)\):

\[ P_{X,Y} : \{(x_i, y_j)\} \rightarrow [0, 1] \]
\[ P_{X,Y}(x_i, y_j) = P(X = x_i \text{ and } Y = y_j). \]
**Example**

- Suppose $X$ and $Y$ have joint probability mass function:

  \[
  f(0,10) = f(0,20) = \frac{2}{18}, \quad f(1,10) = f(1,30) = \frac{3}{18}, \quad f(1,20) = \frac{4}{18}, \quad f(2,30) = \frac{4}{18}.
  \]

- The conditional probabilities are given $\{Y=10\}$:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y=10$</td>
<td>$\frac{2}{18}$</td>
<td>$\frac{3}{18}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$Y=20$</td>
<td>$\frac{2}{18}$</td>
<td>$\frac{4}{18}$</td>
<td>$\frac{3}{18}$</td>
</tr>
<tr>
<td>$Y=30$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{4}{18}$</td>
</tr>
</tbody>
</table>

What is the conditional distribution of $X$ given $\{Y=10\}$?

\[
\begin{align*}
P(X=1, Y=10) &= \frac{3}{18} \\
P(Y=10) &= P(X=0, Y=10) + P(X=1, Y=10) + P(X=2, Y=10) = \frac{2}{18} + \frac{3}{18} + \frac{2}{18} = \frac{5}{18} \\
P(X=1 | Y=10) &= \frac{P(X=1, Y=10)}{P(Y=10)} = \frac{\frac{3}{18}}{\frac{5}{18}} = \frac{3}{5}.
\end{align*}
\]
Basic properties of conditional distribution

**Def:** let $X, Y$ be two discrete R.V. on $\Omega$.
- $P_X$ p.m.f. of $X$.
- $P_Y$ p.m.f. of $Y$.
- $P_{X,Y}$ the joint prob. mass function of $(X,Y)$.

If $P(Y = y) > 0$, then the conditional probability mass function of $X$ given $\{Y = y\}$ is

$$P_{X|Y}(\cdot | y) : \{x_1, x_2, \ldots\} \rightarrow [0,1]$$

$$P_{X|Y}(x; y) \overset{def}{=} \frac{P(X = x; , Y = y)}{P(Y = y)} = P(X = x; | Y = y).$$
For a pair of random variables $X, Y$ if we know the p.m.f. of $Y$ and know the conditional p.m.f. of $X$ given $Y$: $P_{X|Y}(\cdot | \cdot)$

Then we can recover the marginal distribution of $X$ when $X$ takes values in $\{x_1, x_2, \ldots\}$ and $Y$ in $\{y_1, y_2, \ldots\}$

$$P_X(x_i) = P(X = x_i) = \sum_j P(X = x_i, Y = y_j)$$

$$= \sum_{j: P(Y = y_j) > 0} P(X = x_i | Y = y_j) P(Y = y_j) = \sum_{j: P(Y = y_j) > 0} P_{X|Y}(x_i | y_j) P_Y(y_j)$$