Due Friday in class, Jan 26. Relevant sections in Durrett’s textbook 3.2,3.3. Justify all your answers.

1. Let $X_1, X_2, \ldots$ be independent random variables with distribution function $F$. Let 

$$M_n = \max_{1 \leq m \leq n} X_m.$$ 

(a) Suppose $\alpha > 0$ and $F(x) = 1 - x^{-\alpha}$ for $x \geq 1$. Suppose $Y_1$ has distribution function $F_1$, where $F_1(x) = \exp(-x^{-\alpha})$ for all $x > 0$. Show that 

$$\frac{M_n}{n^{1/\alpha}} \Rightarrow Y_1.$$ 

(b) Suppose $\beta > 0$ and $F(x) = 1 - |x|^{\beta}$ for $-1 \leq x \leq 0$. Suppose $Y_2$ has distribution function $F_2$, where $F_2(x) = \exp(-|x|^{\beta})$ for all $x < 0$. Show that 

$$n^{1/\beta}M_n \Rightarrow Y_2.$$ 

(c) Suppose $F(x) = 1 - e^{-x}$ for all $x \geq 0$. Suppose $Y_3$ has distribution function $F_3$, where $F_3(x) = \exp(-e^{-x})$ for all $x \in \mathbb{R}$. Show that 

$$M_n - \log n \Rightarrow Y_3.$$ 

(The distributions of $Y_1, Y_2,$ and $Y_3$ are called the Fréchet, Weibull, and Gumbel distributions. These are called extreme value distributions. It is known that, up to scaling, these are the only distributions that can arise as limits of random variables of the form $(M_n - b_n)/a_n$)

2. Suppose $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ are sequences of random variables, $X$ is a random variable and $c \in \mathbb{R}$ is a constant. Show that if $X_n \Rightarrow X$ and $Y_n \rightarrow c$ in probability, then $X_n Y_n \Rightarrow cX$.

3. Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of probability measures on $\mathbb{R}$ and $F_n$ be the distribution function of $\mu_n$. Let $\mu$ be a measure on $\mathbb{R}$ with $\mu(\mathbb{R}) < 1$, and let $F(x) = \mu((-\infty, x])$.

(a) Show that if for all continuous functions $g : \mathbb{R} \rightarrow \mathbb{R}$ having compact support (meaning there exists $M < \infty$ such that $g(x) = 0$ for any $|x| \geq M$), we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} g(x) d\mu_n(x) = \int_{\mathbb{R}} g(x) d\mu(x),$$

then $\mu_n \Rightarrow \mu$ in vague convergence, that is at continuity points $x, y$ of $F$, $x < y$,

$$\lim_{n \rightarrow \infty} \mu_n((x, y]) = \mu((x, y]).$$
(The converse is also true, but you are not asked to show this.)

(b) Give an example in which \( \mu_n \to \mu \) but the convergence (1) fails for some bounded continuous \( g \).

4. Let \( X \) be a random variable with characteristic function \( \varphi \).
   (a) Show that \( \varphi(t) \in \mathbb{R} \) for all \( t \) if and only if the distribution of \( X \) is symmetric, that is \( X \) and \( -X \) has the same distribution.
   (b) Show that there exists random variables \( Y \) and \( Z \) such that \( \Re(\varphi) \) is the characteristic function of \( Y \) and \( |\varphi|^2 \) is the characteristic function of \( Z \).

5. Let \( X_1, X_2, \ldots \) be i.i.d. random variables with characteristic function \( \varphi \), \( S_n = X_1 + \ldots + X_n \). Show that if \( \varphi'(0) = ia, a \in \mathbb{R} \), then \( S_n/n \to a \) in probability.