Homework 1 Math280B Winter 2018

Due Friday in class, Jan 26. Relevant sections in Durrett's textbook 3.2,3.3. Justify all your answers.

1. Let $X_1, X_2, ...$ be independent random variables with distribution function F . Let

$$M_n = \max_{1 \le m \le n} X_m.$$

(a) Suppose $\alpha > 0$ and $F(x) = 1 - x^{-\alpha}$ for $x \ge 1$. Suppose Y_1 has distribution function F_1, where $F_1(x) = \exp(-x^{-\alpha})$ for all $x > 0$. Show that

$$M_n / n^{1/\alpha} \Rightarrow Y_1.$$

(b) Suppose $\beta > 0$ and $F(x) = 1 - |x|^\beta$ for $-1 \le x \le 0$. Suppose Y_2 has distribution function F_2, where $F_2(x) = \exp(-|x|^\beta)$ for all $x < 0$. Show that n

$$n^{1/\beta} M_n \Rightarrow Y_2.$$

(c) Suppose $F(x) = 1 - e^{-x}$ for all $x \ge 0$. Suppose Y_3 has distribution function F_3, where $F_3(x) = \exp\left(-e^{-x}\right)$ for all $x \in \mathbb{R}$. Show that

$$M_n - \log n \Rightarrow Y_3.$$

(The distributions of Y_1, Y_2, and Y_3 are called the Fréchet, Weibull, and Gumbel distributions. These are called extreme value distributions. It is known that, up to scaling, these are the only distributions that can arise as limits of random variables of the form $(M_n - b_n)/a_n$.)

2. Suppose $(X_n)_{n=1}^\infty$ and $(Y_n)_{n=1}^\infty$ are sequences of random variables, X is a random variable and $c \in \mathbb{R}$ is a constant. Show that if $X_n \Rightarrow X$ and $Y_n \to c$ in probability, then $X_n Y_n \Rightarrow cX$.

3. Let $(\mu_n)_{n=1}^\infty$ be a sequence of probability measures on $\mathbb{R}$ and F_n be the distribution function of μ_n. Let μ be a measure on $\mathbb{R}$ with $\mu(\mathbb{R}) < 1$, and let $F(x) = \mu\left((-\infty, x]\right)$.

(a) Show that if for all continuous functions $g : \mathbb{R} \to \mathbb{R}$ having *compact support* (meaning there exists $M < \infty$ such that $g(x) = 0$ for any $|x| \ge M$), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(x) d\mu_n(x) = \int_{\mathbb{R}} g(x) d\mu(x), \tag{1}$$

then $\mu_n \Rightarrow_v \mu$ in vague convergence, that is at continuity points x, y of F, $x < y$,

$$\lim_{n\to\infty} \mu_n((x, y]) = \mu((x, y]).$$

(The converse is also true, but you are not asked to show this.)

(b) Give an example in which $\mu_n \Rightarrow_v \mu$ but the convergence (1) fails for some bounded continuous g.

4. Let X be a random variable with characteristic function φ.

(a) Show that $\varphi(t) \in \mathbb{R}$ for all t if and only if the distribution of X is symmetric, that is X and $-X$ has the same distribution.

(b) Show that there exists random variables Y and Z such that $Re(\varphi)$ is the characteristic function of Y and $|\varphi|^2$ is the characteristic function of Z.

5. Let $X_1, X_2, \ldots$ be i.i.d. random variables with characteristic function φ, $S_n = X_1 + \ldots + X_n$. Show that if $\varphi'(0) = ia$, $a \in \mathbb{R}$, then $S_n/n \to a$ in probability.