Math 280 A Homework 1

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Exercise 1. One example is setting $\omega = \{1, 2, 3, 4\}$, and the subset collection as $\{\emptyset, \omega, \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}\}$. One may easily show that this collection satisfies the conditions, but $\{1, 2\} \cap \{1, 3\} = \{1\}$ which is out of the collection.

Exercise 2. (i) Denote the right hand side of the equation as $\mathcal{A}$. First show $\mathcal{A} \subset \sigma(\mathcal{C})$. This is done by seeing that every element in $\mathcal{A}$ is constructed by either taking countably many unions of elements in $\mathcal{C}$, or take one more step of complement (if the element's complement is countable).

Then we show $\sigma(\mathcal{C}) \subset \mathcal{A}$. This is done by showing $\mathcal{A}$ is a σ-algebra. It is easy to check the empty set and whole set is in $\mathcal{A}$, and the closeness of complement is clear in the definition. The closeness of countable unions is shown here. Assume $A_i \in \mathcal{A}, i \in \mathbb{N}^+$, if all A_i are countable, clearly the union is countable thus $\cup_i A_i \in \mathcal{A}$. If one of the A_i satisfies A_i^c countable, by $(\cup_i A_i)^c \subset A_i^c$ we know $(\cup_i A_i)^c$ countable, which indicates $\cup_i A_i \in \mathcal{A}$.

(ii) In (i) we have shown that $\mathcal{A}$ is a σ-algebra, and it is easy to see that $P(\cdot)$ is between 0 and 1, $P(\emptyset) = 0$, $P(\Omega) = 1$. It is remaining to show the countably additivity.

Assume A_i are disjoint sets in $\mathcal{A}$, if all of them are countable, the union is countable thus$\sum_i P(A_i) = P(\cup_i A_i) = 0$. If one of them is uncountable, the union is uncountable, and $\sum_i P(A_i) = P(\cup_i A_i) = 1$. It is not possible that there are two uncountable A_i and A_j since $A_i \subset A_j^c$.

Exercise 3. Borel sets are countably generated by open sets, which is countably generated by open balls. Consider 'rational balls' with rational centers (center located with all indexes rational on all axes) and rational radius. Since rational numbers are countable, these balls are countable.

The final shot is generate an arbitrary open ball by countably operations on 'rational balls'.

Exercise 4. (i) We omit the proof of closeness of complement and inclusion of empty set and Ω. The closeness of finite union can be inducted by showing closeness of one union. Assume $A, B \in \cup_i \mathcal{F}_i$, then there exists i, j such that $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$. Denote $k = \max(i, j)$, and we have $A, B \in \mathcal{F}_k$ by $\mathcal{F}_i, \mathcal{F}_j \subset \mathcal{F}_k$. $\mathcal{F}_k$ is a σ-algebra, so $A \cup B \in \mathcal{F}_k$.

(ii) One counterexample is as follows. Denote $\Omega = \mathbb{N}$, $\mathcal{F}_i = \{A : A \in 2^{[i]} or A^c \in 2^{[i]}\}$ with $[i] = \{1, 2, \dots, i\}$ and $2^{\cdot}$ as power set. Clearly the sequence satisfies the requirement, and one can construct the set of even numbers by countably many operations. But the set of even numbers in not in $\mathcal{F}_i$.

Exercise 5. (i)(iii) is yes and the proof is trivial.

(ii) is no. Consider A as even numbers, and B as follows. B takes even numbers between 2^m and 2^{m+1} when m is even, and take odd numbers when m is odd. Clearly both A and B has asymptotic density of

1/2, but $A \cup B$ has no asymptotic density. One may show that if m is even, when $n = 2^m$ the ratio is close to 5/6, and if m is odd, the ratio is close to 2/3 at $n = 2^m$.

(iv) An counterexample is taking union of singletons which construct the sequence in (ii) which has no asymptotic density. Each singleton has asymptotic density 0 thus is in the set, but the union is not.

Exercise 6. We begin with set $\mathcal{C} = \{\text{all open intervals in } \mathbb{R}\}$. Clearly this is a π system. Denote

$$\mathcal{L} = \{A : \forall \epsilon, \exists B \text{ such that } P(A \triangle B) < \epsilon \text{ with } B \text{ as finite union of intervals}\}$$

If we show that $\mathcal{L}$ is a λ system, by π-λ theorem we know that $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}$ which finish the proof.

It is not hard to show the first two properties. We show the third property here. For $A_i \in \mathcal{L}$ with $A_i \subset A_{i+1}$ for all i, we need $\cup_i A_i \in \mathcal{L}$.

For ϵ, observe that $P(\cup_i A_i \setminus A_k) \to 0$ as $k \to \infty$ (Monotone convergence theorem, continuity of measure, Lebesgue's bounded convergence theorem, etc), so we may choose N such that $P(\cup_i A_i \setminus A_N) < \epsilon/2$. Now since $A_N \in \mathcal{L}$, choose B such that $P(A_N \triangle B) < \epsilon/2$ and B is a finite union of intervals. Now

$$P(\cup_i A_i \triangle B) \leq P(A_N \triangle B) + P(\cup_i A_i \triangle A_N) = P(A_N \triangle B) + P(\cup_i A_i \setminus A_N) = \epsilon.$$

This completes the proof.