Math 280 A Homework 2

October 15, 2017

Exercise 1. Denote

$$\mathcal{L} = \{B : \text{ There exists some countable subclass } \mathcal{C}_B \subset \mathcal{C} \text{ such that } B \in \sigma(\mathcal{C}_B)\}.$$

The upshot is to show that $\mathcal{L}$ is a σ-algebra. This is done by (i) show that $\mathcal{L}$ is a λ-system, and (ii) show that $\mathcal{L}$ contains a π system which containing $\mathcal{C}$.

(i) It is easier to use an alternative definition of λ-system which includes (1) inclusion of $\emptyset, \Omega$; (2) closed in complement; (3) closed in disjoint countable union, (1)(2) are easy, we show (3) here. Assume $B_i \in \mathcal{L}$, and B_i are disjoint. Assume $\mathcal{C}_{B_i}$ are corresponding countable subclasses such that $B_i \in \sigma(\mathcal{C}_{B_i})$. Consider $\sigma(\cup)_i\mathcal{C}_{B_i})$. Since $\mathcal{C}_{B_i}$ are countable, $\cup_i\mathcal{C}_{B_i}$ is countable. And it is easy to see for all i, $B_i \in \sigma(\cup_i\mathcal{C}_{B_i})$. Therefore $\cup_i B_i \in \sigma(\cup_i\mathcal{C}_{B_i})$ by closeness of countable unions of σ-algebra.

(ii) Consider the class $\mathcal{A}$ such that contains all the finite intersections of elements in $\mathcal{C}$. Clearly this is a π-system. We need to show $\mathcal{A} \subset \mathcal{L}$. Assume $A \in \mathcal{A}$ such that $A = \cap_{i=i}^n A_i$ such that $A_i \in \mathcal{C}$. Clearly $A \in \sigma(A_1, \ldots, A_n)$ so $A \in \mathcal{L}$. This finishes (ii).

The final shot is by $\pi - \lambda$ theorem, $\sigma(\mathcal{C}) \subset \sigma(\mathcal{A}) \subset \mathcal{L}$.

Exercise 2. (i) Assume A is an atom, we need to show that A is a single point. This is done by shrinking sequentially towards an single point. First divide $\mathbb{R}$ to $(n, n+1]$ for $n \in \mathbb{Z}$. By the definition of atom, there is one n such that $P(A \cap (n, n+1]) = P(A)$. Now divide the interval $(n, n+1]$ into two, only one will contain the probability mass. Keep doing this operation and denote the interval that contains the probability mass as I_n in the n-th round. It is easy to see that $|I_n| = 2^{-n}$, and $P(I_n \cap A) = P(A)$. Now by continuity of measure, $\lim_n P(I_n \cap A) = P(\lim I_n \cap A) = P(A)$.

Now consider $\lim I_n \cap A$. This set is non-empty (positive probability) and contains only one point (easy to show). And it is also trivial to see that this point (say x) satisfies $F(x) - F(x-) > 0$.

(ii) If $P(A \cap B) \neq 0$, then $P(A \setminus B) = P(A \setminus (A \cap B)) = 0$. Similarly $P(B \setminus A) = 0$. Therefore $P(A \triangle B) = P(A \setminus B) + P(B \setminus A) = 0$.

(iii) If $\{x\}$ is an atom, x is corresponding to an interval $(a, b) \subset [0, 1]$ where $a = F(x-)$ and $b = F(x)$. Since $b - a > 0$, we may find a rational number q between a and b and make a map from x to q. Such an map is an injection from the set of atoms towards the set of rational numbers in $[0, 1]$. Therefore the set of atoms has to be countable.

Exercise 3. (i) If all A_i are finite, the union is finite thus both sides are 0. If one of them is infinite, then the union is infinite thus both sides are 1. More than two A_is being infinite and disjoint is impossible since it will definitely merge in the tail of $\mathbb{N}$.

(ii) One counterexample will be $A_i = \{i, i+1, \ldots\}$. $A_i \downarrow \emptyset$ but $\mu(A_i)$ stays at 1.

Exercise 4. (i) Denote the original definition as P_1, and the alternative definition as P_2. Clearly $P_1 \geq P_2$ since $\sum_i P(B_i) \geq P(\cup_i B_i)$, and $\cup_i B_i \in \mathcal{F}$.

The reverse is shown as follows. For any $\epsilon > 0$, and any A, there exists some $B \in \mathcal{F}$ such that $P_2(A) > P(B) - \epsilon$. Since P_1 and P_2 agrees on $\mathcal{F}$, there exists some $B_i \in \mathcal{F}_0$ such that $P(B) > \sum_i P(B_i) - \epsilon$. Note that $A \subset B$, thus $A \subset \cup_i B_i$. Therefore $P_2(A) > P_1(A) - 2\epsilon$. Since ϵ is arbitrary, it concludes that $P_2 = P_1$.

To show that there is an set achieving the inf, for any A there exists $B_n \in \mathcal{F}$ such that $P(B_n) - 1/n \leq P^*(A) \leq P(B)$. Then consider $\cap_n B_n$. Clearly $A \subset \cap_n B_n$ and $P(\cap_n B_n) - 1/n \leq P^*(A)$ for any n. Therefore $P(\cap_n B_n) = P^*(A)$, which is what we need.

(ii) Note that the inner measures are achievable as well (just think about the complements). The only if side is direct. The if side is done by realizing that there exists $B_1, B_2 \in \mathcal{F}$ such that $B_2 \subset A \subset B_1$ and $P(B_2) = P_*(A) = P^*(A) = P(B_1)$. Therefore A is B_2 union some null set (sets with measure 0). Since all null sets are in $\mathcal{F}$ (use definition to see that), $A \in \mathcal{F}$ is illustrated.

Exercise 5. Denote $\{x : f(x) \leq a\}$ as $A(a)$. For a sequence $y_i \in A$, and $y_i \to x$. By the definition of l.s.c, $f(x) \leq \liminf f(y) \leq a$, which shows that $x \in A$, thus A closed. Now assume $f(x) > \liminf f(y)$, then there exists a subsequence of $y_{i_j} \to x$ such that $f(y) \to c < f(x)$. This shows that $A(c)$ is not closed.

Note that f maps (a, ∞) to an open set, and collections of sets like (a, ∞) generates the σ-algebra. Therefore f is measurable.