Solution sketch to 280B HW problems

1 Homework 1

1. Let $X_1, X_2, \ldots$ be independent random variables with distribution function F. Let

$$M_n = \max_{1 \le m \le n} X_m.$$

(a) Suppose $\alpha > 0$ and $F(x) = 1 - x^{-\alpha}$ for $x \ge 1$. Suppose Y_1 has distribution function F_1, where $F_1(x) = \exp(-x^{-\alpha})$ for all $x > 0$. Show that

$$M_n/n^{1/\alpha} \Rightarrow Y_1.$$

(b) Suppose $\beta > 0$ and $F(x) = 1 - |x|^\beta$ for $-1 \le x \le 0$. Suppose Y_2 has distribution function F_2, where $F_2(x) = \exp(-|x|^\beta)$ for all $x < 0$. Show that n

$$n^{1/\beta} M_n \Rightarrow Y_2.$$

(c) Suppose $F(x) = 1 - e^{-x}$ for all $x \ge 0$. Suppose Y_3 has distribution function F_3, where $F_3(x) = \exp\left(-e^{-x}\right)$ for all $x \in \mathbb{R}$. Show that

$$M_n - \log n \Rightarrow Y_3.$$

Solution:

Note that $\mathbb{P}\left(M_n \le x\right) = \mathbb{P}(X_1 \le x)^n = F(x)^n$. Plug in the given expression for $F(x)$ and pass to limit when $n \to \infty$.

2. Suppose $(X_n)_{n=1}^\infty$ and $(Y_n)_{n=1}^\infty$ are sequences of random variables, X is a random variable and $c \in \mathbb{R}$ is a constant. Show that if $X_n \Rightarrow X$ and $Y_n \to c$ in probability, then $X_n Y_n \Rightarrow cX$.

Solution:

Case (i): $c > 0$. Take any $\epsilon > 0$ any small constant , we have

$$\mathbb{P}\left(X_n Y_n \le cx\right) \le \mathbb{P}\left(Y_n \le c(1-\epsilon)\right) + \mathbb{P}\left(X_n \le \frac{x}{1-\epsilon}\right).$$

Using these $\epsilon > 0$ such that $x/(1-\epsilon)$ is a continuity point of F_X and send n to ∞, we obtain

$$\limsup_{n\to\infty} \mathbb{P}\left(X_n Y_n \le cx\right) \le F_X\left(\frac{x}{1-\epsilon}\right).$$

In the other direction,

$$\mathbb{P}\left(X_n Y_n \le cx\right) \ge \mathbb{P}\left(X_n \le \frac{x}{1+\epsilon'}\right) - \mathbb{P}\left(Y_n \ge c(1+\epsilon') \text{ or } Y_n \le 0\right).$$

Choose $\epsilon' > 0$ such that $x/(1+\epsilon')$ is a continuity point of F_X, we have

$$\liminf_{n\to\infty} \mathbb{P}\left(X_n Y_n \le cx\right) \ge F_X\left(\frac{x}{1+\epsilon'}\right).$$

Combine these two inequalities, we conclude that at continuity point x of F_X, $\lim_{n\to\infty}\mathbb{P}(X_nY_n \le cx) = F_X(x) = F_{cX}(cx)$.

The case $c<0$ follows from taking $-Y_n$.

Case (ii): $c=0$, we need to show that $X_nY_n \Rightarrow 0$, that is $\mathbb{P}(|X_nY_n|\ge\epsilon)\to 0$. We have for any $M>0$,

$$\mathbb{P}(|X_nY_n|\ge\epsilon) \le \mathbb{P}(|Y_n|\ge 1/M) + \mathbb{P}(|X_n|\ge M\epsilon).$$

Choose M such that $\pm M\epsilon$ are continuity point of F_X , we have $\limsup_{n\to\infty}\mathbb{P}(|X_nY_n|\ge\epsilon)\le F_X(-M\epsilon)+1-F_X(M\epsilon)$. Send M to infinity we obtain the statement.

3. Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of probability measures on $\mathbb{R}$ and F_n be the distribution function of μ_n. Let μ be a measure on $\mathbb{R}$ with $\mu(\mathbb{R})<1$, and let $F(x)=\mu((-\infty,x])$.

(a) Show that if for all continuous functions $g:\mathbb{R}\to\mathbb{R}$ having *compact support* (meaning there exists $M<\infty$ such that $g(x)=0$ for any $|x|\ge M$), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}} g(x)d\mu_n(x) = \int_{\mathbb{R}} g(x)d\mu(x), \tag{1}$$

then $\mu_n \Rightarrow_v \mu$ in vague convergence, that is at continuity points x,y of F, $x<y$,

$$\lim_{n\to\infty}\mu_n((x,y]) = \mu((x,y]).$$

(The converse is also true, but you are not asked to show this.)

(b) Give an example in which $\mu_n \Rightarrow_v \mu$ but the convergence (1) fails for some bounded continuous g.

Solution:

(a) The same argument as in Theorem 3.2.3 in the textbook, applied to indicator $\mathbf{1}_{(x,y]}$ which results in continuous functions of compact support.

(b) Choose an example that fails to be tight.

4. Let X be a random variable with characteristic function φ.

(a) Show that $\varphi(t)\in\mathbb{R}$ for all t if and only if the distribution of X is symmetric, that is X and $-X$ has the same distribution.

(b) Show that there exists random variables Y and Z such that $Re(\varphi)$ is the characteristic function of Y and $|\varphi|^2$ is the characteristic function of Z.

Solution:

(a) if direction: since X and $-X$ have the same distribution, $\varphi(t)=\mathbb{E}\left[e^{itX}\right]=\frac{1}{2}\mathbb{E}\left[e^{itX}+e^{-itX}\right]$, which is real. Only if direction: use the inversion formula to show for example $\mathbb{P}(X\ge x)=\mathbb{P}(X\le -x)$, $x>0$ a continuity point.

(b) Let B be a random variable such that $\mathbb{P}(B=1)=\mathbb{P}(B=-1)=1/2$ and B is independent of X. Then $Y=BX$ has ch.f. $Re(\varphi)$.

Let X_1,X_2 be two independent random variables with the same distribution as X. Then the ch.f. of X_1-X_2 is

$$\mathbb{E}\left[e^{itX_1-X_2}\right]=\mathbb{E}\left[e^{itX_1}\right]\mathbb{E}\left[e^{-itX_2}\right]=\varphi(t)\varphi(-t)=|\varphi(t)|^2.$$

5. Let $X_1,X_2,\ldots$ be i.i.d. random variables with characteristic function φ, $S_n=X_1+\ldots+X_n$. Show that if $\varphi'(0)=ia$, $a\in\mathbb{R}$, then $S_n/n\to a$ in probability.

Solution:

It suffices to show $S_n/n\Rightarrow a$.

$$\mathbb{E}\left[e^{itS_n/n}\right]=\varphi(t/n)^n=\left(\varphi(0)+\varphi'(0)\frac{t}{n}+o(\frac{t}{n})\right)^n\to e^{\varphi'(0)t} \text{ as } n\to\infty.$$

The Dirac mass at a has ch.f. e^{iat}, the claim follows.

2 Homework 2

1. Suppose $X_1, X_2, \dots$ are i.i.d. random variables such that $\mathbb{E}[X_m] = 0$ and $E[X_m^2] = \sigma^2$ for all m, where $0 < \sigma^2 < \infty$. Show that

$$\sum_{m=1}^{n} X_m / \left(\sum_{m=1}^{n} X_m^2 \right)^{1/2} \Rightarrow Z.$$

Solution:
By SLLN,

$$\frac{1}{n} \sum_{m=1}^{n} X_m^2 \to \sigma^2 \ a.s.$$

The claim follows from CLT that $\frac{1}{\sqrt{n}\sigma} \sum_{m=1}^{n} X_m \Rightarrow Z$, $\sqrt{n}\sigma / \left(\sum_{m=1}^{n} X_m^2\right)^{1/2} \to 1$ a.s. and Problem 2 in HW1.

2. For $n \in \mathbb{N}$, suppose X_n has the binomial distribution with parameters n and p_n. This means that $X_n = \xi_{n,1} + \ldots + \xi_{n,n}$, where $\xi_{n,j}$, $1 \le j \le n$, are i.i.d. and $\mathbb{P}(\xi_{n,i} = 1) = p_n$ and $\mathbb{P}(\xi_{n,i} = 0) = 1 - p_n$.

(a) Show that if $\lim_{n\to\infty} np_n(1 - p_n) = \infty$, then

$$\frac{X_n - np_n}{\sqrt{np_n(1 - p_n)}} \Rightarrow Z.$$

(b) Show that if $\lim_{n\to\infty} np_n = \lambda \in (0, \infty)$, then $X_n \Rightarrow \text{Poisson}(\lambda)$.

Solution:
(a) check the conditions of Lindeberg-Feller Theorem 3.4.5. (b) Poisson approximation theorem 3.6.1.

3. Suppose $X_1, X_2, ...$ are independent. Suppose $\mathbb{P}(X_m = -m) = \mathbb{P}(X_m = m) = m^{-2}/2$ for all $m \ge 1$ and $\mathbb{P}(X_m = -1) = \mathbb{P}(X_m = 1) = (1 - m^{-2})/2$ for $m \ge 2$. Let $S_n = X_1 + \ldots + X_n$. Show that $\text{var}(S_n)/n \to 2$ but $S_n/\sqrt{n} \Rightarrow Z$.

Solution:
For variance, $\text{var}(X_m) = m^2 \cdot m^{-2} + 1 - m^{-2} = 2 - m^{-2}$, by independence

$$\text{var}(S_n) = \sum_{m=1}^{n} \text{var}(X_m) = 2n - \sum_{m=1}^{n} m^{-2}.$$

The summation converges to a constant when $n \to \infty$.
Write down the ch.f. of X_m,

$$\begin{aligned}\varphi_m(t) &= \frac{1}{2m^2}\left(e^{itm} + e^{-itm}\right) + \frac{1}{2}\left(1 - \frac{1}{m^2}\right)\left(e^{it} + e^{-it}\right) \\ &= m^{-2}\cos(tm) + (1 - m^{-2})\cos(t).\end{aligned}$$

For $S_n/\sqrt{n}$ its ch.f. is

$$\prod_{m=1}^{n} \varphi_m\left(t/\sqrt{n}\right) = \prod_{m=1}^{n} \left(m^{-2}\cos\left(\frac{tm}{\sqrt{n}}\right) + (1 - m^{-2})\cos\left(\frac{t}{\sqrt{n}}\right)\right).$$

For small m,

$$m^{-2}\cos\left(\frac{tm}{\sqrt{n}}\right) + (1 - m^{-2})\cos\left(\frac{t}{\sqrt{n}}\right) = 1 - \frac{t^2}{n} + \frac{t^2}{2nm^2} + o(t^2m^2/n);$$

for large m, consider $m^{-2}\cos\left(\frac{tm}{\sqrt{n}}\right)+(1-m^{-2})\cos\left(\frac{t}{\sqrt{n}}\right)$ as $m^{-2}\left[\cos\left(\frac{tm}{\sqrt{n}}\right)-\cos\left(\frac{t}{\sqrt{n}}\right)\right]+\cos\left(\frac{t}{\sqrt{n}}\right)$, a perturbation of $\cos\left(\frac{t}{\sqrt{n}}\right)$ with error bounded by $2/m^2$. Divide the product at some L_n, say $L_n = n^\alpha$ with $\alpha < 1/2$, use the two types of asymptotics.

4. Suppose X and Y are random variables such that $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$, and suppose $\mathcal{G}$ is a σ-field. Show that if $\mathbb{E}[Y|\mathcal{G}] = X$ and $\mathbb{E}[Y^2|\mathcal{G}] = X^2$, then $X = Y$ a.s.

Solution:

From the assumption that $\mathbb{E}[Y|\mathcal{G}] = X$ we know that X is measurable with respect to $\mathcal{G}$. Then

$$\begin{aligned}\mathbb{E}\left[(Y-X)^2|\mathcal{G}\right] &= \mathbb{E}\left[Y^2|\mathcal{G}\right] - 2\mathbb{E}\left[XY|\mathcal{G}\right] + \mathbb{E}\left[X^2|\mathcal{G}\right] \\ &= \mathbb{E}\left[Y^2|\mathcal{G}\right] - 2X\mathbb{E}[Y|\mathcal{G}] + X^2.\end{aligned}$$

Plug in the assumptions we obtain that $\mathbb{E}\left[(Y-X)^2|\mathcal{G}\right] = 0$ a.s. Therefore $\mathbb{E}\left[(Y-X)^2\right] = 0$, it follows that $Y - X = 0$ a.s.

5. If $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{G}$ are σ-fields, we say $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent given $\mathcal{G}$ if $\mathbb{P}(A\cap B|\mathcal{G}) = \mathbb{P}(A|\mathcal{G})\mathbb{P}(B|\mathcal{G})$ for all $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Prove that if $\mathbb{P}(A|\sigma(\mathcal{F}_2,\mathcal{G})) = \mathbb{P}(A|\mathcal{G})$ for all $A \in \mathcal{F}_1$, where $\sigma(\mathcal{F}_2,\mathcal{G})$ denotes the σ-field generated by $\mathcal{F}_2$ and $\mathcal{G}$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent given $\mathcal{G}$.

Solution:

Let $A \in \mathcal{F}_1,\ B \in \mathcal{F}_2$. Then by the tower property, $\mathbb{E}\left[\mathbf{1}_A\mathbf{1}_B|\mathcal{G}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_A\mathbf{1}_B|\sigma(\mathcal{G},\mathcal{F}_2)\right]|\mathcal{G}\right]$. From the assumption $\mathbb{P}(A|\sigma(\mathcal{F}_2,\mathcal{G})) = \mathbb{P}(A|\mathcal{G})$,

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_A\mathbf{1}_B|\sigma(\mathcal{G},\mathcal{F}_2)\right] &= \mathbf{1}_B\mathbb{E}\left[\mathbf{1}_A|\sigma(\mathcal{G},\mathcal{F}_2)\right] \\ &= \mathbf{1}_B\mathbb{E}\left[\mathbf{1}_A|\mathcal{G}\right].\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_A\mathbf{1}_B|\mathcal{G}\right] &= \mathbb{E}\left[\mathbf{1}_B\mathbb{E}\left[\mathbf{1}_A|\mathcal{G}\right]|\mathcal{G}\right] \\ &= \mathbb{E}\left[\mathbf{1}_A|\mathcal{G}\right]\mathbb{E}\left[\mathbf{1}_B|\mathcal{G}\right],\end{aligned}$$

which means they are conditionally independent given $\mathcal{G}$.

6. Suppose $f : \mathbb{R}^2 \to [0,\infty)$ is a measurable function, and X and Y are random variables with joint density f . Let $g(x) = \int_{\mathbb{R}} f(x,y)dy$, and for simplicity assume $g(x) > 0$ for all $x \in \mathbb{R}$. Let $h(x,y) = f(x,y)/g(x)$. Now for $\omega \in \Omega$ and $B \in \mathcal{B}(\mathbb{R})$, let

$$Q(\omega, B) = \int_B h(X(\omega), y)dy.$$

(a) Show that g is a density for X.

(b) Show that Q is a regular conditional distribution for Y given $\sigma(X)$.

Solution:

(a) note that $\mathbb{P}(X \le x) = \mathbb{P}(X \le x, Y \in \mathbb{R}) = \int_{(-\infty,x]\times\mathbb{R}} f(x,y)dxdy$, then g is a density for X by Fubini theorem.

(b) check the definition of r.c.d.:

(i) for each B, $\omega \to Q(\omega,B)$ is a version of $\mathbb{P}(Y \in B|X)$. To see this, for any set in $\sigma(X)$, it is of the form $X^{-1}(A)$ for some Borel set A,

$$\mathbb{P}(Y\in B, X\in A) = \int_B\int_A f(x,y)dxdy = \int_A g(x)\int_B h(x,y)dydx = \mathbb{E}\left[\mathbf{1}_{\{X\in A\}}\int_B h(X,y)dy\right].$$

(ii) for any given ω, $h(X(\omega,y)) = f(X(\omega),y)/g(X(\omega))$ is a density function.

3 Homework 3

1. Suppose $(X_n)_{n=0}^{\infty}$ and $(Y_n)_{n=0}^{\infty}$ are martingales with respect to filtration $(\mathcal{F}_n)_{n=0}^{\infty}$. Assume $\mathbb{E}[X_n^2] < \infty$ and $\mathbb{E}[Y_n^2] < \infty$ for all n.

(a) Show that for all n,

$$\mathbb{E}[X_n Y_n - X_0 Y_0] = \sum_{m=1}^{n} \mathbb{E}\left[(X_m - X_{m-1})(Y_m - Y_{m-1})\right].$$

(b) Show that if $X_0 = 0$, then for all n,

$$\mathbb{E}[X_n^2] = \sum_{m=1}^{n} \mathbb{E}[(X_m - X_{m-1})^2].$$

Solution:

(a) by the martingale property check that $\mathbb{E}[X_m Y_m - X_{m-1}Y_{m-1}] = \mathbb{E}\left[(X_m - X_{m-1})(Y_m - Y_{m-1})\right]$, then sum up. (b) follows from (a) by taking $Y_n = X_n$.

2. Let $(S_n)_{n=0}^{\infty}$ be a simple random walk. That is, $S_0 = 0$ and, for $n \geq 1$, we have $S_n = \xi_1 + \ldots + \xi_n$, where $\xi_1, \xi_2, \ldots$ are i.i.d. with $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = 1/2$.

(a) Show that $S_n^2 - n$ is a martingale.

(b) Let $T = \inf\{n : \; S_n \notin (-a, a)\}$,where $a \in \mathbb{N}$. Show that $\mathbb{E}[T] = a^2$.

Solution:

(a) $\mathbb{E}\left[S_n^2 - n | \mathcal{F}_{n-1}\right] = \mathbb{E}\left[(S_{n-1} + X_n)^2 - n | \mathcal{F}_{n-1}\right] = S_{n-1}^2 - n - 2S_{n-1}\mathbb{E}[X_n|\mathcal{F}_{n-1}] + \mathbb{E}[X_n^2|\mathcal{F}_{n-1}]$. Since X_n is independent of $\mathcal{F}_{n-1}$, we have $\mathbb{E}[X_n|\mathcal{F}_{n-1}] = \mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^2|\mathcal{F}_{n-1}] = \mathbb{E}[X_n^2] = 1$.

(b) Write $Y_n = S_n^2 - n$ and consider the martingale $Y_{n \wedge T}$. From $\mathbb{E}Y_{n\wedge T} = 0$ we have $\mathbb{E}[S_{n\wedge T}^2] = \mathbb{E}[T \wedge n]$. By bounded convergence theorem, $\lim_{n\to\infty} \mathbb{E}[S_{n\wedge T}^2] = \mathbb{E}[S_T^2] = a^2$. By monotone convergence theorem $\mathbb{E}[T] = \lim_{n\to\infty} \mathbb{E}[T \wedge n] = a^2$.

3. Let $a > 0$, and let $X_1, X_2, \ldots$ be i.i.d. random variables having a normal distribution with mean $\mu > 0$ and variance 1. Let $S_0 = a$, and let $S_n = a + X_1 + \cdots + X_n$ for $n \in \mathbb{N}$.

(a) Let $Y_n = e^{-2\mu S_n}$. Show that $(Y_n)_{n=0}^{\infty}$ is a martingale.

(b) Show that

$$\mathbb{P}\left(S_n \leq 0 \text{ for some } n\right) \leq e^{-2\mu a}.$$

Solution:

(a) check that $\mathbb{E}[e^{-2\mu X_n}] = 1$ by plugging the density function of X_n.

(b) $\{S_n \leq 0\} = \{Y_n \geq 1\}$. Apply Doob's maximal inequality Theorem 5.4.2.

4. Let $(X_n)_{n=0}^{\infty}$ be a submartingale such that $X_0 = 0$ and

$$\sup_{n\geq 0} X_n(\omega) < \infty$$

for all $\omega \in \Omega$. Let $\xi_n = X_n - X_{n-1}$, and suppose $\mathbb{E}[\sup_{n\geq 0} \xi_n^+] < \infty$. Show that $(X_n)_{n=0}^{\infty}$ converges a.s.

Solution:

Follow the reasoning of Theorem 5.3.1. Let $0 < K < \infty$ and let $N = \inf\{n : \; X_n \geq K\}$. Consider $X_{n\wedge N} - K$, it is a submartingale satisfying the property that

$$(X_{n\wedge N} - K)^+ \leq \sup_{1\leq j\leq n} \xi_j^+,$$

therefore

$$\sup \mathbb{E}\left(X_{n\wedge N}-K\right)^{+} \leq \mathbb{E}\sup_{n} \xi_n^{+} < \infty.$$

By the martingale convergence theorem 5.2.8. $\lim X_n$ exists on the event $N=\infty$. Letting $K \to \infty$, limit exists on the event $\{\limsup X_n < \infty\}$, which is assumed.

5. Let $(X_n)_{n=0}^{\infty}$ be a martingale such that $X_0 = 0$ and $|X_{n+1} - X_n| \leq r$ for all n, where r is a positive real number.

(a) Show that

$$\mathbb{E}\left[\max_{0\leq m\leq n} X_m^2\right] \leq 4r^2 n.$$

(b) Show that if $x > 0$, then

$$\mathbb{P}\left(\max_{0\leq m\leq n} |X_m| > x\sqrt{n}\right) \leq \frac{r^2}{x^2}.$$

Solution:

(a) Note that $\mathbb{E}[X_n^2] = \sum_{j=1}^{n} \mathbb{E}\left[\left(X_j - X_{j-1}\right)^2\right] \leq r^2 n$. The statement follows from Doob's L^2-maximal inequality.

(b) Follows from Doob's maximal inequality 5.4.2 applied to the submartingale X_n^2.

6. Let $(\mathcal{F}_n)_{n=0}^{\infty}$ be a filtration, and let $\mathcal{F}_\infty = \sigma(\cup_{n=0}^{\infty}\mathcal{F}_n)$. Let A be an event such that $A \in \mathcal{F}_\infty$ but A is independent of $\mathcal{F}_0$. Suppose $\mathbb{P}(A) = 1/2$. Let $X_n = \mathbb{P}\left(A|\mathcal{F}_n\right)$ for all $n \geq 0$.

(a) Show that if $\frac{1}{2} \leq x \leq 1$, then

$$\mathbb{P}\left(\sup_{n\geq 0} X_n \geq x\right) \leq \frac{1}{2x}.$$

(b) Show that

$$\frac{3}{4} \leq \mathbb{E}\left[\sup_{n\geq 0} X_n\right] \leq \frac{1+\log 2}{2}.$$

Solution:

(a) X_n is a non-negative martingale, $\mathbb{E}X_n = \mathbb{P}(A) = 1/2$. By Doob's maximal inequality

$$x\mathbb{P}(\max_{0\leq j\leq n} X_j \geq x) \leq \mathbb{E}X_n = 1/2.$$

Let $n \to \infty$ we obtain (a).

(b) Upper bound follows from integrating the tail bound in (a).

By Levy $0-1$ law $\lim_{n\to\infty} X_n = \mathbf{1}_A$ a.s. Note that since A is independent of $\mathcal{F}_0$, X_0 is constant $1/2$. Then

$$\begin{aligned}\mathbb{E}\left[\sup_{n\geq 0} X_n\right] &\geq \mathbb{E}\left[\max\left\{X_0, \lim_{n} X_n\right\}\right] \\ &= \mathbb{E}[\max\{1/2, \mathbf{1}_A\}] = 1/2\mathbb{P}(A^c) + \mathbb{P}(A) = 3/4.\end{aligned}$$