Math 280 A Homework 3

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Exercise 1. (i) We show that the set of $\mathcal{F}$-measurable functions is closed in taking limits. It suffices to show that the set is closed in taking limsup. Note that

$$\{x : \limsup f_i(x) \leq c\} = \cap_{j\geq 1} \cup_{n\geq 1} \cap_{k\geq n} \{x : f_k(x) \leq c + 1/j\},$$

and $\{x : f_k(x) \leq c + 1/j\}$ is in $\mathcal{F}$, which is suffice to show the demanded result.

(ii) Assume f is $\mathcal{F}$-measurable. First decompose f into positive functions f^+ and f^-, which represents the positive and negative parts of f. Then set $f_n^+ = 2^{-n}\lfloor 2^n f^+ \rfloor \wedge n$. We see that f_n^+ is simple, $\mathcal{F}$-measurable and converging to f^+. Thus f^+ is $\mathcal{F}$-measurable. Similarly f^- is $\mathcal{F}$-measurable which completes the proof.

Exercise 2. (i) Borel sets are countably generated (proved before). Then just show that $X^{-1}(\cdot)$ operation preserves the complement and countable unions.

(ii) If side is straight forward by checking the preimage. Only if side we consider $Y = \lim_i Y_i$ where Y_i are simple functions on $\sigma(X)$. It is easy to see the simple functions on $\sigma(X)$ are measurable functions of X ($\mathbb{1}(A) = \mathbb{1}(X \in B)$ where $A = X^{-1}(B)$), therefore the limit of simple functions is also a measurable function of X.

Exercise 3. Using Markov inequality:

$$P(X \geq 1/n) \leq nE(X) = 0.$$

Therefore

$$P(X \neq 0) = P(\cup_n \{X \geq 1/n\}) \leq \sum_n P(X \geq 1/n) = 0,$$

which finishes the proof.

Exercise 4. (i) The critical point is to check the countable additivity (since other properties are easy to check). Assume $\{A_i\}$ are disjoint sets, and $A = \cup_i A_i$. By monotone convergence theorem:

$$\begin{aligned}\nu(A) = \int f \mathbb{1}_A f d\mu &= \int (\lim_n \sum_{i=1}^n \mathbb{1}_{A_i}) f d\mu \\ &= \lim_n \int (\sum_{i=1}^n \mathbb{1}_{A_i}) f d\mu = \lim_n \sum_{i=1}^n \int \mathbb{1}_{A_i} f d\mu = \sum_i \nu(A_i)\end{aligned}$$

(ii) Denote $E_n = \{\omega : f(\omega) \geq n\}$. By the fact that $\int f d\mu < \infty$, we have $\nu(E_n) = \int_{E_n} f d\mu \to 0$. Therefore choose N such that $\nu(E_N) < 0.5\varepsilon$. Now since

$$\nu(A) = \nu(A \cap E_N) + \nu(A \setminus E_N) \leq \nu(E_N) + \int_{A\setminus E_N} f d\mu \leq 0.5\varepsilon + N\mu(A),$$

choose $\delta = \varepsilon/2N$ and the desired result is obtained.

Exercise 5. In order to use DCT we need to show that $\sum_i |f_i|$ is integrable. This is done by monotone convergence theorem:

$$\int \sum_i |f_i| d\mu = \int \lim_n \sum_{i=1}^n |f_i| d\mu = \lim_n \sum_{i=1}^n \int |f_i| d\mu = \sum_i \int |f_i| d\mu < \infty.$$

Therefore since $|\sum_{i=1}^n f_i| \leq \sum_{i=1}^n |f_i| \leq \sum_i |f_i|$, we use DCT:

$$\sum_i \int f_i d\mu = \lim_n \sum_{i=1}^n \int f_i d\mu = \lim_n \int \sum_{i=1}^n f_i d\mu = \int \lim_n \sum_{i=1}^n f_i d\mu = \int \sum_i f_i d\mu,$$

as desired.