Exercise 1. (i) We show that the set of $\mathcal{F}$-measurable functions is closed in taking limits. It suffices to show that the set is closed in taking limsup. Note that
\[ \{x : \limsup_i f_i(x) \leq c\} = \cap_{j \geq 1} \cup_{n \geq 1} \cap_{k \geq n} \{x : f_k(x) \leq c + 1/j\}, \]
and $\{x : f_k(x) \leq c + 1/j\}$ is in $\mathcal{F}$, which is suffice to show the demanded result.

(ii) Assume $f$ is $\mathcal{F}$-measurable. First decompose $f$ into positive functions $f^+$ and $f^-$, which represents the positive and negative parts of $f$. Then set $f^+_n = 2^{-n} \lfloor 2^n f^+ \rfloor \wedge n$. We see that $f^+_n$ is simple, $\mathcal{F}$-measurable and converging to $f^+$. Thus $f^+$ is $\mathcal{F}$-measurable. Similarly $f^-$ is $\mathcal{F}$-measurable which completes the proof.

Exercise 2. (i) Borel sets are countably generated (proved before). Then just show that $X^{-1}(\cdot)$ operation preserves the complement and countable unions.

(ii) If side is straight forward by checking the preimage. Only if side we consider $Y = \lim_i Y_i$ where $Y_i$ are simple functions on $\sigma(X)$. It is easy to see the simple functions on $\sigma(X)$ are measurable functions of $X$ ($\mathbb{I}(A) = \mathbb{I}(X \in B)$ where $A = X^{-1}(B)$), therefore the limit of simple functions is also a measurable function of $X$.

Exercise 3. Using Markov inequality:
\[ P(X \geq 1/n) \leq nE(X) = 0. \]
Therefore
\[ P(X \neq 0) = P(\cup_n \{X \geq 1/n\}) \leq \sum_n P(X \geq 1/n) = 0, \]
which finishes the proof.

Exercise 4. (i) The critical point is to check the countable additivity (since other properties are easy to check). Assume $\{A_i\}$ are disjoint sets, and $A = \cup_i A_i$. By monotone convergence theorem:
\[ \nu(A) = \int f \mathbb{1}_A f d\mu = \int (\lim_n \sum_{i=1}^n \mathbb{1}_{A_i}) f d\mu \]
\[ = \lim_n \int (\sum_{i=1}^n \mathbb{1}_{A_i}) f d\mu = \lim_n \sum_{i=1}^n \int \mathbb{1}_{A_i} f d\mu = \sum_i \nu(A_i) \]

(ii) Denote $E_n = \{\omega : f(\omega) \geq n\}$. By the fact that $\int f d\mu < \infty$, we have $\nu(E_n) = \int_{E_n} f d\mu \to 0$. Therefore choose $N$ such that $\nu(E_N) < 0.5\varepsilon$. Now since
\[ \nu(A) = \nu(A \cap E_N) + \nu(A \setminus E_N) \leq \nu(E_N) + \int_{A \setminus E_N} f d\mu \leq 0.5\varepsilon + N\mu(A), \]
choose $\delta = \varepsilon/2N$ and the desired result is obtained.
**Exercise 5.** In order to use DCT we need to show that $\sum |f_i|$ is integrable. This is done by monotone convergence theorem:

$$\int \sum_i |f_i| \, d\mu = \int \lim_{n \to \infty} \sum_{i=1}^{n} |f_i| \, d\mu = \lim_{n \to \infty} \sum_{i=1}^{n} \int |f_i| \, d\mu = \sum_{i} \int |f_i| \, d\mu < \infty.$$ 

Therefore since $|\sum_{i=1}^{n} f_i| \leq \sum_{i=1}^{n} |f_i| \leq \sum_{i} |f_i|$, we use DCT:

$$\sum_{i} \int f_i \, d\mu = \lim_{n} \sum_{i=1}^{n} \int f_i \, d\mu = \lim_{n} \sum_{i=1}^{n} f_i \, d\mu = \int \lim_{n} \sum_{i=1}^{n} f_i \, d\mu = \int \sum_{i} f_i \, d\mu,$$

as desired.