Math 280 A Homework 4

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Exercise 1. Realize that

$$E(X) = E(X\mathbb{1}(X > 0))$$

and apply Cauchy-Schwartz inequality:

$$E(X)^2 = E(X\mathbb{1}(X > 0))^2 \leq E(X^2)E((\mathbb{1}(X > 0))^2) = E(X^2)P(X > 0).$$

Exercise 2. Start from g as indicator function (which is easy), and extend to g as simple functions. Now assume $g \geq 0$. Use a sequence of simple functions g_n to approach g pointwisely from below, and use MCT on both sides. Then for general g, consider decomposition $g = g^+ - g^-$ where both functions of the right hand side are non-negative. From $\int |g(x)|\mu(dx) < \infty$ we may have both $\int g^+(x)\mu(dx)$ and $\int g^-(x)\mu(dx)$ finite. So we can do the subtraction and finish the work.

Exercise 3. Consider a 4 element space $\Omega = \{a, b, c, d\}$ with each element weights with probability 1/4. Now consider $\mathcal{A}_1 = \{\{a, b\}, \{b, c\}\}$ and $\mathcal{A}_1 = \{\{a, c\}, \{b, d\}\}$. It is easy to check the independence but $\sigma(\mathcal{A}_1)$ contains $\{a\}$ which is not independent of $\{b, d\} \in \mathcal{A}_2$.

Exercise 4. First we show Y_n are distributed as described. First extend $[0, 1]$ to $[0, 2^n]$ and see that for $\omega * 2^n \in [2k, 2k + 1)$ interval, $Y(\omega) = 1$. Clearly these ω takes up half of the intervals (since intervals like $[2k, 2k + 1)$ takes up half portion in $[0, 2^n]$. Therefore $P(Y_n = 1) = 0.5$.

Then we show the independence. This is equivalent to show that for a finite collection of integers $n_1, \ldots, n_k$ and binaries $i_1, \ldots, i_k \in \{0, 1\}$, we have $P(Y_{n_1} = i_1, \ldots, Y_{n_k} = i_k) = \prod_j P(Y_{n_j} = i_j)$. Without loss of generality we assume n_j are monotone increasing, and this can be done by induction.

WLOG, set all $i_j = 1$. For Y_{n_1}, find out the intervals such that $Y_{n_1} = 1$ and realize that inside each interval, there are half portion that makes $Y_{n_2} = 1$. Induct this to the general case and the proof is complete.

Exercise 5. (i) WLOG assume $E(Y) = 0$. Assume $P(Y \neq 0) > 0$. Then there exists some n and set A such that $Y > 1/n$ on A, with $P(A) > 0$. Then

$$P(X \in A, Y > 1/n) = P(X \in A)P(Y > 1/n)$$

by independence. However $\{X \in A\}$ implies $Y > 1/n$ therefore the LHS equals $P(X \in A)$. Since $P(A) \neq 0$, we have $P(Y > 1/n) = 1$ which contradicts with $E(Y) = 0$.

(ii) Take $Y = \mathbb{1}(X \geq 1/2)$. The critical point is that by observing Z only, one piece of X is missing and that is whether X is at the left or the right side of $1/2$.