Math 280 A Homework 5

November 6, 2017

Exercise 1. By Fubini, we have

$$\int F(x+c) - F(x)dx = \int\int \mathbb{1}_{(x,x+c]}dFdx = \int\int \mathbb{1}_{(x,x+c]}dxdF = \int cdF = c.$$

Exercise 2. Denote $\mathcal{F}_1, \mathcal{F}_2$ as σ-algebras on Ω_1, Ω_2 respectively. We would like to prove $\mathcal{F} = \mathcal{F}_0$, where $\mathcal{F}$ as the σ-algebra generated by rectangle sets and denote $\mathcal{F}_0$ as the smallest σ-algebra that makes π_1, π_2 measurable. Note that $\mathcal{F}$ is the smallest σ-algebra that contains $\mathcal{A} = \{E_1 \times E_2, E_1 \in \mathcal{F}_1, E_2 \in \mathcal{F}_2\}$. Take π_1 as an example, the preimage of $E_1 \in \mathcal{F}_1$, is $E_1 \times \Omega_2$. Therefore, $\mathcal{F}_0$ is the smallest σ-algebra that contains $\mathcal{B} = \{E_1 \times \Omega_2, E_1 \in \mathcal{F}_1\} \cup \{\Omega_1 \times E_2, E_2 \in \mathcal{F}_2\}$.

Notice that $\mathcal{A}$ contains $\mathcal{B}$, and any element in $\mathcal{A}$ is an intersection of two elements in $\mathcal{B}$, therefore $\mathcal{F} = \mathcal{F}_0$ by the standard inclusion argument.

Exercise 3. (i) By Fubini's theorem:

$$\begin{aligned}\int_{(a,b]} F(y) - F(a)dG(y) &= \int\int \mathbb{1}_{(a,b]}(y)\mathbb{1}_{(a,y]}(x)dF(x)dG(y) \\ &= \int_{\mathbb{R}^2} \mathbb{1}_{a<x\le y\le b}(x,y)d(\mu\times\nu) \\ &= (\mu\times\nu)(a<x\le y\le b)\end{aligned}$$

(ii) By part (i) we have

$$\int_{(a,b]} F(y)dG(y) = F(a)G(b) - F(a)G(a) + (\mu\times\nu)(a<x\le y\le b).$$

By symmetry we have

$$\int_{(a,b]} G(y)dG(y) = G(a)F(b) - F(a)G(a) + (\mu\times\nu)(a<y\le x\le b).$$

Add up both sides, notice that the addition of the last part gives measure $\mu\times\nu$ on the rectangle $(a,b]\times(a,b]$, plus the diagonal $\{(x,x), a<x\le b\}$. The rectangle part will give $(F(b)-F(a))(G(b)-G(a))$. Denote A as the set of discontinuities inside $(a,b]$ of F, we have

$$\begin{aligned}(\mu\times\nu)((x,x), a<x\le b) &= \int_{(a,b]} \mathbb{1}_{\{y\}}(x)dF(x)dG(y) \\ &= \int \mu(\{y\})dG(y) = \int_A \mu(\{y\})dG(y) + \int_{A^c} \mu(\{y\})dG(y)\end{aligned}$$

The first part is a summation of countably many points, and the second part is 0 due to the integrand is 0. Now by MCT:

$$\int_A \mu(\{y\})dG(y) = \int_{(a,b]} \sum_i \mathbb{1}_{\{a_i\}}\mu(\{y\})dG(y) = \sum_i \int \mathbb{1}_{\{a_i\}}\mu(\{y\})dG(y) = \sum_i \mu(\{a_i\})\nu(\{a_i\}).$$

Combine the equations and the result follows.

(iii) The desired probability is

$$P(X_1 \leq X_2) = \int P(X_1 \leq x)dF(x) = \int F(x)dF(x).$$

Send $a \to -\infty$, $b \to \infty$ in the result of (ii), and the result follows.

Exercise 4. If $P(A_i \quad i.o.) = 1$, then for all elements in set $E = \{A_i \quad i.o.\}$, there is some A_i containing this element, therefore it is easy to see that $P(\cup_i A_i) = 1$. On the other side, if $P(\cup_i A_i) = 1$, $P(\cap_i A_i^c) = 0$. By independence we have $\prod_{i=1}^{\infty} P(A_i^c) = 0$. Notice that all $P(A_i^c) > 0$ by the fact that $P(A_i) < 1$, then for all positive integer k we have $\prod_{i=k}^{\infty} P(A_i^c) = 0$. Now

$$P\big(\cap_{i=1}^{\infty} \cup_{n\geq i} A_n\big) = 1 - P\big(\cup_{i=1}^{\infty} \cap_{n\geq i} A_n^c\big) = 1 - \lim_i \prod_{n\geq i} P(A_i^c) = 1$$

If we allow $P(A_i) = 1$, just choose $A_1 = \Omega$ and all other $A_i = \emptyset$ and the relation is violated.

Exercise 5. Denote event $A_i = \{|X_n|/c_n > 1/n\}$. If $P(A_i \quad i.o.) = 0$, for almost every ω, $X_n/c_n \to 0$ is guaranteed. By Borel-Cantelli Lemma, choose c_n such that $P(|X_n|/c_n > 1/n) < 1/n^2$. Now $\sum_n P(A_n) < \infty$, and $P(A_i \quad i.o.) = 0$ is ensured