Math 280 A Homework 6

November 14, 2017

Exercise 1. (I) Assume the subsequence $X_{n_k}$ satisfies $\lim_k E(X_{n_k}) = \liminf_n E(X_n)$. Now since $X_{n_k}$ converges in probability to $X$ as well, there is a further subsequence $X_{n_{k_j}}$ such that converges to $X$ almost surely. Therefore by Fatou’s Lemma, $\lim_j E X_{n_{k_j}} \geq E(X)$ which is just $\liminf_n E(X_n) \geq E(X)$ since $\lim_j E X_{n_{k_j}} = \lim_k E(X_{n_k})$.

(II) Apply the same procedure to the sequences $Y + X_n$ and $Y - X_n$ and we will have $\liminf_n E(Y - X_n) \geq E(Y - X)$ and $\liminf_n E(Y + X_n) \geq E(Y + X)$, which is just $\liminf_n E(X_n) \geq E(X) \geq \limsup_n E(X_n)$. This is enough to show the demanded result.

Exercise 2. (I) The right hand side is

$$E \left( \frac{|X - Y|(1 + |Y - Z|) + |Y - Z|(1 + |X - Y|)}{(1 + |X - Y|)(1 + |Y - Z|)} \right) \geq E \left( \frac{|X - Y| + |Y - Z| + |X - Y||Y - Z|}{1 + |X - Y| + |Y - Z| + |X - Y||Y - Z|} \right)$$

Now since $|X - Y| + |Y - Z| + |X - Y||Y - Z| \geq |X - Z|$, It suffices to show the function $f(x) = x/(1 + x)$ monotone increasing, which is trivial by taking one order derivative.

(II) When $X_n$ converges in probability to 0, we assume for any $\epsilon > 0$ we have eventually $P(|X_n| \geq \epsilon) \leq \epsilon$. Now denote $A = \{|X_n| \geq \epsilon\}$, we have

$$E(|X_n|/(1 + |X_n|)) = E(|X_n|/(1 + |X_n|)I_A) + E(|X_n|/(1 + |X_n|)1_{A^c}) \leq E(1_A) + \frac{\epsilon}{1 + \epsilon} \leq 2\epsilon$$

since $\epsilon$ is arbitrary we have $d(X_n, 0) \to 0$.

On the other hand, if $d(X_n, 0) \to 0$, eventually we will have $d(X_n, 0) \leq \epsilon^2$. Now

$$\epsilon^2 \geq E(|X_n|/(1 + |X_n|)) \geq E(|X_n|/(1 + |X_n|)1_A) \geq \frac{\epsilon}{1 + \epsilon} P(A)$$

which implies $P(|X_n| \geq \epsilon) \leq \epsilon(\epsilon + 1)$. This is equivalent to converging in probability to zero.

Exercise 3. It suffices to show that converging in probability implies converging almost surely. Suppose it is not true, then there exists some $\omega$ such that $X_n(\omega)$ does not converge to $X(\omega)$ with $P(\omega) = \delta > 0$. WLOG assume $\limsup_n X_n(\omega) - X(\omega) = \epsilon > 0$. Now consider the subsequence $X_{n_k}$ such that achieves the limsup. Consider the sequence $P(|X_{n_k} - X| \geq \epsilon/2)$. By the fact that the subsequence converging to $X$ in probability, this sequence should converge to 0, but on $\omega$ it will eventually differ by more than $\epsilon/2$ so the sequence will eventually be no less than $\delta$ which is a contradiction.

Exercise 4. Denote $B_n = \{X_n > \lambda_n\}$, and $P(B_n) = 1 - F(\lambda_n)$. Notice that $B_n$ implies $A_n$, therefore $B_n \ i.o.$ implies $A_n \ i.o.$ By Borel-Cantelli Lemma, $\sum_n (1 - F(\lambda_n)) = \infty$ implies $P(B_n \ i.o.) = 1$ which implies $P(A_n \ i.o.) = 1$. 

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Now we show the reverse. Note that by Borel-Cantelli Lemma, \( \sum_n (1 - F(\lambda_n)) < \infty \) implies \( P(B_n \text{ i.o.}) = 0 \). Note that \( \sum_n (1 - F(\lambda_n)) < \infty \) means that \( \lambda_n \) reaches the right support edge of \( X_n \), which means for any realization \( x \), eventually \( \lambda_n > x \) will happen. Now for \( \omega \notin \{B_n \text{ i.o.}\} \), there exists some \( N \) such that \( X_n(\omega) \leq \lambda_n \) for all \( n \geq N \). Assume \( M = \max_{1 \leq n \leq N} X_n \), and there is some \( N' \) such that \( \lambda_n > M \) for all \( n \geq N' \). Now we see that for all \( \lambda_n \) with \( n \geq \max(N, N') \), it is no less than \( M \), thus no less than all \( X_n \) before \( N \). For those \( X_i \) with \( i \) between \( N \) and \( N' \) (if there are), \( X_i \) are controlled by \( \lambda_i \), and since \( \lambda_i \) is increasing, all are less than the last one, \( \lambda_n \). Therefore \( \omega \notin \{A_n \text{ i.o.}\} \). Now \( P(\{\{B_n \text{ i.o.}\}^c) = 1 \), therefore \( P(\{\{A_n \text{ i.o.}\}^c = 1 \), or \( P(\{A_n \text{ i.o.}\}^c = 0 \).

**Exercise 5.** \( X_n \) converges in \( L^2 \) if and only if the sequence in Cauchy in \( L^2 \), or (WLOG, \( m < n \))

\[
\lim_{m,n \to \infty} \int |X_m - X_n|^2 dP = \sum_{i=m+1}^{n} a_i^2 C \to 0
\]

which means that \( S_n = \sum_{i=1}^{n} a_i^2 \) sequence is Cauchy as well, therefore \( S_n \) converges, or \( \lim_n S_n = \sum_{i=1}^{\infty} a_i < \infty \).

**Exercise 6.** Use the inequality

\[
|X_n + Y_n|1_{|X_n + Y_n| \geq 2K} \leq 2X_n 1_{|X_n| \geq K} + 2Y_n 1_{|Y_n| \geq K}
\]

and we can easily control the right hand side by the definition of the u.i. Therefore the sequence \( \{X_n + Y_n\} \) is u.i.

The inequality is from the fact that if \( |X_n + Y_n| \geq 2K \), the one with larger absolute value must go beyond \( K \), and the whole \( |X_n + Y_n| \) is less than double of that larger absolute value.