Math 280 A Homework 6

November 14, 2017

Exercise 1. (I) Assume the subsequence X_{n_k} satisfies $\lim_k E(X_{n_k}) = \liminf_n E(X_n)$. Now since X_{n_k} converges in probability to X as well, there is a further subsequence $X_{n_{k_j}}$ such that converges to X almost surely. Therefore by Fatou's Lemma, $\lim_j EX_{n_{k_j}} \geq E(X)$ which is just $\liminf_n E(X_n) \geq E(X)$ since $\lim_j EX_{n_{k_j}} = \lim_k E(X_{n_k})$.

(II) Apply the same procedure to the sequences $Y + X_n$ and $Y - X_n$ and we will have $\liminf_n E(Y - X_n) \geq E(Y - X)$ and $\liminf_n E(Y + X_n) \geq E(Y + X)$, which is just $\liminf_n E(X_n) \geq E(X) \geq \limsup_n E(X_n)$. This is enough to show the demanded result.

Exercise 2. (I) The right hand side is

$$E(\frac{|X-Y|(1+|Y-Z|)+|Y-Z|(1+|X-Y|}{(1+|X-Y|)(1+|Y-Z|)}) \geq E(\frac{|X-Y|+|Y-Z|+|X-Y||Y-Z|}{1+|X-Y|+|Y-Z|+|X-Y||Y-Z|})$$

Now since $|X-Y|+|Y-Z|+|X-Y||Y-Z| \geq |X-Z|$, It suffices to show the function $f(x) = x/(1+x)$ monotone increasing, which is trivial by taking one order derivative.

(II) When X_n converges in probability to 0, we assume for any $\varepsilon > 0$ we have eventually $P(|X_n| \geq \varepsilon) \leq \varepsilon$. Now denote $A = \{|X_n| \geq \varepsilon\}$, we have

$$E(|X_n|/(1+|X_n|)) = E(|X_n|/(1+|X_n|)\mathbb{1}_A) + E(|X_n|/(1+|X_n|)1_{A^c}) \leq E(\mathbb{1}_A) + \frac{\varepsilon}{1+\varepsilon} \leq 2\varepsilon$$

since ε is arbitrary we have $d(X_n, 0) \to 0$.

On the other hand, if $d(X_n, 0) \to 0$, eventually we will have $d(X_n, 0) \leq \varepsilon^2$. Now

$$\varepsilon^2 \geq E(|X_n|/(1+|X_n|)) \geq E(|X_n|/(1+|X_n|)\mathbb{1}_A) \geq \frac{\varepsilon}{1+\varepsilon}P(A)$$

which implies $P(|X_n| \geq \varepsilon) \leq \varepsilon(\varepsilon+1)$. This is equivalent to converging in probability to zero.

Exercise 3. It suffices to show that converging in probability implies converging almost surely. Suppose it is not true, then there exists some ω such that $X_n(\omega)$ does not converge to $X(\omega)$ with $P(\omega) = \delta > 0$. WLOG assume $\limsup_n X_n(\omega) - X(\omega) = \varepsilon > 0$. Now consider the subsequence X_{n_k} such that achieves the limsup. Consider the sequence $P(|X_{n_k} - X| \geq \varepsilon/2)$. By the fact that the subsequence converging to X in probability, this sequence should converge to 0, but on ω it will eventually differ by more than $\varepsilon/2$ so the sequence will eventually be no less than δ which is a contradiction.

Exercise 4. Denote $B_n = \{X_n > \lambda_n\}$, and $P(B_n) = 1 - F(\lambda_n)$. Notice that B_n implies A_n, therefore $B_n \quad i.o.$ implies $A_n \quad i.o.$. By Borel-Cantelli Lemma, $\sum_n (1 - F(\lambda_n)) = \infty$ implies $P(B_n \quad i.o.) = 1$ which implies $P(A_n \quad i.o.) = 1$.

Now we show the reverse. Note that by Borel-Cantelli Lemma, $\sum_n (1-F(\lambda_n)) < \infty$ implies $P(B_n \quad i.o.) = 0$. Note that $\sum_n (1-F(\lambda_n)) < \infty$ means that λ_n reaches the right support edge of X_n, which means for any realization x, eventually $\lambda_n > x$ will happen. Now for $\omega \notin \{B_n \quad i.o.\}$, there exists some N such that $X_n(\omega) \leq \lambda_n$ for all $n \geq N$. Assume $M = \max_{1 \leq n \leq N} X_n$, and there is some N' such that $\lambda_n > M$ for all $n \geq N'$. Now we see that for all λ_n with $n \geq \max(N, N')$, it is no less than M, thus no less than all X_n before N. For those X_i with i between N and N' (if there are), X_i are controlled by λ_i, and since λ_i is increasing, all are less than the last one, λ_n. Therefore $\omega \notin \{A_n \quad i.o.\}$. Now $P(\{B_n \quad i.o.\}^c) = 1$, therefore $P(\{A_n \quad i.o.\}^c) = 1$, or $P(\{A_n \quad i.o.\}) = 0$.

Exercise 5. X_n converges in L^2 if and only if the sequence in Cauchy in L^2, or (WLOG, $m < n$)

$$\lim_{m,n\to\infty} \int |X_m - X_n|^2 dP = \sum_{i=m+1}^{n} a_i^2 C \to 0$$

which means that $S_n = \sum_{i=1}^n a_i^2$ sequence is Cauchy as well, therefore S_n converges, or $\lim_n S_n = \sum_{i=1}^\infty a_i < \infty$.

Exercise 6. Use the inequality

$$|X_n + Y_n| \mathbb{1}_{|X_n+Y_n| \geq 2K} \leq 2X_n \mathbb{1}_{|X_n| \geq K} + 2Y_n \mathbb{1}_{|Y_n| \geq K}$$

and we can easily control the right hand side by the definition of the u.i. Therefore the sequence $\{X_n + Y_n\}$ is u.i..

The inequality is from the fact that if $|X_n + Y_n| \geq 2K$, the one with larger absolute value must go beyond K, and the whole $|X_n + Y_n|$ is less than double of that larger absolute value.