Math 280 A Homework 7

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Exercise 1. It is clear that $(X - X_n) \leq X$ therefore $(X - X_n)_+ \leq X$, which means $E((X - X_n)_+ \mathbb{1}((X - X_n)_+ \geq M)) \leq E(X \mathbb{1}(X \geq M))$. This indicates that $(X - X_n)_+$ is u.i. since $E(X) < \infty$. Also $X_n \to X$ in probability implies $(X - X_n)_+ \to 0$ in probability. Therefore $(X - X_n)_+ \to 0$ in L_1. Now realize that

$$E|X_n - X| = E((X - X_n)_+) + E((X - X_n)_-) = 2E((X - X_n)_+) - E(X_n - X)$$

The right hand side goes to zero, which proves what desired.

Exercise 2. (I) It is trivial to show the unboundedness of $E(|X_i|)$, as $\sum_{k=1}^{K} 1/k \log k \approx \log(\log K) \to \infty$.

(II) We will construct triangular array to apply the weak law of large numbers in triangular arrays. Denote $X_{i,j} = X_j \mathbb{1}(|X_j| \leq i)$. To apply the theorem, we need to check: (1) $nP(|X_i| \geq n) \to 0$. This is because $nP(|X_i| \geq n) = n \sum_{k=n}^{\infty} 1/k^2 \log k \leq n/\log n \sum_{k=n}^{\infty} 1/k^2 \approx 1/\log n \to 0$. (2) $nE(X_{i,n}^2)/n^2 \to 0$. This is equivalent to $\frac{1}{n} \sum_{k=2}^{n} 1/\log n \leq [\log(\log n)]^2/n \to 0$. Now apply the weak law of large number of triangular array we see that $S_n/n \to \mu$ in probability where μ is defined as $\sum_{k=2}^{\infty} (-1)^k/k \log k$.

Exercise 3. (I) It suffices to show that $M(s)/s \to 0$. This is done by rewrite $M(s) = \int_0^s (F(s) - F(t))dt = sF(s) - \int_0^s F(t)dt$. The first part gives 1 when divided by s. The second part also gives one since we can have some N such that $F(N) \geq 1 - \epsilon$. Then take $N' = N^2$ and $\int_0^{N^2} F(t)dt/N^2 \geq N(N-1)(1-\epsilon)/N^2 \approx 1 - \epsilon$, so the limit is 1.

(II) Just check the conditions for weak law of large numbers for triangular array. The first condition is given. The second one is

$$nE(X_i^2 \mathbb{1}(X_i^2 \leq b_n))/b_n^2 \to 0.$$

First realize that $b_n = nM(n)$ always holds, since function $f(s) = s/M(s)$ is right continuous and only jumps down when there is a discontinuous point. Therefore the junction point of f and n is always a solid point, which means the equality always holds. Now since $jP(X_1 > b_j) \to 0$, there is some N such that for all $n \geq N$, $P(X_1 > b_n) \leq \varepsilon/n$. Consider another large integer T, and use the fact that $E(X_i^2 \mathbb{1}(X_i^2 \leq b_n)) \leq \int_0^{b_n} 2yP(X_1 > y)dy \approx \sum_n 2P(X_1 > b_n)(b_{n+1}^2 - b_n^2)$, we have when $n = TN$:

$$TNE(X_i^2 \mathbb{1}(X_i^2 \leq b_{TN})) \leq \frac{2TN \sum_{n \leq N} P(X_1 > b_n)(b_{n+1}^2 - b_n^2)}{b_{TN}^2} + \frac{2TN \sum_{n=N+1}^{TN} P(X_1 > b_n)(b_{n+1}^2 - b_n^2)}{b_{TN}^2}$$

The first part is less than $2TNb_N^2/b_{TN}^2$ and by plugging in $b_n = nM(n)$, we have the first part controlled by $2N/T$. The second part is controlled by plugging in $P(X_1 > b_n) \leq \varepsilon/n$, which is

$$\frac{2TN \sum_{n=N+1}^{TN} b_j^2/j(j+1)}{b_{TN}^2} \leq \frac{2\varepsilon \sum_{n=N+1}^{TN} M^2(b_j)}{TNM^2(b_{TN})} \leq 2\varepsilon.$$

Taking $T = 2N/\varepsilon$ and we will have the whole thing controlled by 4ε, which is what we need.

Now apply the weak law of large numbers and by using $b_n = nM(n)$ the result follows.

Exercise 4. First realize that $Y_n = X_n/|X_{n-1}|$ is IID with uniform distribution on the unit ball. And then $\log|X_n| = \sum_{i=1}^{n} \log|Y_n|$. In order to use the strong law of large numbers we need to specify the distribution of $|Y_n|$. In fact:

$$P(|Y_n| \leq x) = P(|X_1| \leq x) = \pi x^2/\pi = x^2$$

Therefore the density for $|Y_i|$ is $f(y) = 2y$, and the expectation of Y_i is $\int_0^1 \log y \cdot 2y dy = -1/2$. So $\log|X_n|/n \to -1/2$ almost surely.

Exercise 5. Divide the upper and lower side by n, and the lower side goes to EX_1 almost surely by strong law of large numbers. Therefore it suffices to show that almost surely $\max_{i\leq n}|X_i|/n \to 0$. To see this, let $\lambda_n = \varepsilon n$, and observe that $\sum_n P(|X_n| > \lambda_n) = \sum_n P(|X_i|/\varepsilon > n) \approx E(|X_i|/\varepsilon) < \infty$. By Borel-Cantelli Lemma, this implies $P(|X_n| > \lambda_n \quad i.o.) = 0$. By a previous exercise, this also implies $P(\max_{i\leq n}|X_i| > \lambda_n \quad i.o.) = 0$. Therefore $\limsup_n[\max_{i\leq n}|X_i|/n] \leq \varepsilon$ almost surely for arbitrary ε, which is $\max_{i\leq n}|X_i|/n \to 0$ with probability one.