Math 280 A Homework 8

December 4, 2017

Exercise 1. Denote X_i as the return random variable as $X_i = 1+a$ with probability $1/2+q$, and $X_i = 1-a$ with probability $1/2-q$. The target is to maximize $Y = \prod_i X_i$ which is the aggregate return of the bet. It is equivalent to maximize $\log Y = \sum_i \log X_i$. Using SLLN, the expression $\log Y/n$ is very close to the quantity $E(\log X_1)$, which by calculation, is maximized when $a = 2q$.

Exercise 2. First if $\sum_i X_i < \infty$ a.s., by the three series theorem it is easy to have these two statements.

Reversely, by $\sum_i P(X_i > c) < \infty$ and Borel Cantelli Lemma, we know a.s. X_i will be no larger than c. Therefore $\sum_i X_i < \infty$ a.s. is equivalent to $\sum_i X_i \mathbb{1}(X_i \le c) < \infty$ a.s.. Now by MCT, $\sum_i E(X_i \mathbb{1}(X_i \le c)) = E(\sum_i X_i \mathbb{1}(X_i \le c)) < \infty$, which indicates $\sum_i X_i \mathbb{1}(X_i \le c) < \infty$ a.s..

Exercise 3. We aim to use three series theorem to show the convergence. First: $\infty > \sum_n E(|X_n| \mathbb{1}(|X_n| > 1)) \ge \sum_n P(|X_n| > 1)$, so the first condition checked. Then by the fact that $\sum_n E(X_n) = 0 = \sum_n E(X_n \mathbb{1}(|X_n| \le 1)) + \sum_n E(X_n \mathbb{1}(|X_n| > 1))$, and the second part is controlled by $\sum_n E(|X_n| \mathbb{1}(|X_n| > 1))$ which is finite, therefore $\sum_n E(X_n \mathbb{1}(|X_n| \le 1))$ is also finite. So second condition checked. The third condition is done by combining the fact that $\sum_n (E(X_n \mathbb{1}(|X_n| \le 1)))^2 \le \sum_n E(|X_n| \mathbb{1}(|X_n| \le 1)) < \infty$ and $\sum_n E(X_n^2 \mathbb{1}(|X_n| \le 1)) < \infty$.

Exercise 4. For all positive α, the first two conditions are easily checked. The third condition is satisfied if and only if $\sum_n E(X_n^2 \mathbb{1}(|X_n| \le 1)) < \infty$. And it is easily checked that this holds if and only if $\alpha > 1/2$.

Exercise 5. Realize that $E((\log |X_1|)_+) < \infty$ if and only if for any $c > 0$, $E((\log |X_1|)_+/c) < \infty$. Note that $E((\log |X_1|)_+/c) < \infty$ if and only if $\sum_n P((\log |X_1|)/c > n) = \sum_n P((\log |X_n|) > nc) < \infty$. By Borel Cantelli Lemma, for any $s < 1$, if $\sum_n P((\log |X_n|) > nc) = \infty$, take $c = -\log s$ and a.s. $|X_n| > s^{-n}$ will happen infinitely many times, which means $\sum_n |X_n| s^n$ will contain infinitely many numbers larger than one, which is infinite. When $\sum_n P((\log |X_n|) > nc) < \infty$, take $c = -\log s/2$, and a.s. the sequence $\sum_n |X_n| s^n$ will be controlled by $\sum_n [\sqrt{s}]^n$ after some N, which indicates that $\sum_n |X_n| s^n < \infty$.