Math 140A: “Winter” 2016
Homework 5

Available	Friday, January 29	Due	Friday, February 5

Turn in the homework by 5:00pm on Friday, February 5, in the homework box in the basement of AP&M. Late homework will not be accepted.

1. Let (a_n) be a sequence in $\mathbb{R}$ (not necessarily bounded), and let

$$A = \{x \in \mathbb{R} \colon a_n \geq x \text{ for all sufficiently large } n\}.$$

Show that $\sup A = \liminf_{n\to\infty} a_n$. [*Hint*: consider separately the cases when (a_n) is bounded below or not. For the second case, recall the convention that $\sup(\varnothing) = -\infty$.]

2. Let (a_n) be a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} a_n = +\infty$. Show that $a_n > 0$ for all sufficiently large n, and $\lim_{n\to\infty} \dfrac{1}{a_n} = 0$.

3. Let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and let $\theta \in \mathbb{R}$.

(a) Show that the matrix exponential $e^{\theta J}$ has the form

$$e^{\theta J} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos\theta I + \sin\theta J.$$

(b) Using part **(a)** and our construction of $\mathbb{C}$, show that every complex number z has a representation of the form $z = |z|e^{i\theta}$ for some $\theta \in [0, 2\pi)$, and this θ is unique if $z \neq 0$. (The angle θ is called the **argument** of z, $\theta = \mathrm{Arg}(z)$.)

4. **(a)** Show that there are exactly two complex numbers z so that $z^2 = -1$: $z = i$ and $z = -i$.

(b) Show that the complex conjugation function $\mathfrak{c} \colon \mathbb{C} \to \mathbb{C}$, $\mathfrak{c}(z) = \overline{z}$, is a field isomorphism.

(c) It is a fact that any field isomorphism $\varphi \colon \mathbb{C} \to \mathbb{C}$ must act as the identity on $\mathbb{R}$: for $x \in \mathbb{R}$, $\varphi(x) = x$; you do not need to prove this. Using this fact and part **(a)**, show that the *only* field isomorphisms $\varphi \colon \mathbb{C} \to \mathbb{C}$ are $\varphi(z) = z$ and $\varphi(z) = \mathfrak{c}(z) = \overline{z}$.