Turn in the homework by 5:00pm on Friday, February 5, in the homework box in the basement of AP&M. Late homework will not be accepted.

1. Let \((a_n)\) be a sequence in \(\mathbb{R}\) (not necessarily bounded), and let 
\[ A = \{ x \in \mathbb{R} : a_n \geq x \text{ for all sufficiently large } n \}. \]
Show that \(\sup A = \lim \inf_{n \to \infty} a_n\). [Hint: consider separately the cases when \((a_n)\) is bounded below or not. For the second case, recall the convention that \(\sup(\emptyset) = -\infty\).]

2. Let \((a_n)\) be a sequence in \(\mathbb{R}\) with \(\lim_{n \to \infty} a_n = +\infty\). Show that \(a_n > 0\) for all sufficiently large \(n\), and \(\lim_{n \to \infty} \frac{1}{a_n} = 0\).

3. Let \(J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\) and let \(\theta \in \mathbb{R}\).
   (a) Show that the matrix exponential \(e^{\theta J}\) has the form 
   \[ e^{\theta J} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \cos \theta I + \sin \theta J. \]
   (b) Using part (a) and our construction of \(\mathbb{C}\), show that every complex number \(z\) has a representation of the form \(z = |z|e^{i\theta}\) for some \(\theta \in [0, 2\pi)\), and this \(\theta\) is unique if \(z \neq 0\). (The angle \(\theta\) is called the argument of \(z\), \(\theta = \text{Arg}(z)\).)

4. (a) Show that there are exactly two complex numbers \(z\) so that \(z^2 = -1\): \(z = i\) and \(z = -i\).
   (b) Show that the complex conjugation function \(\overline{\cdot} : \mathbb{C} \to \mathbb{C}, \overline{z} = \overline{z}\), is a field isomorphism.
   (c) It is a fact that any field isomorphism \(\varphi : \mathbb{C} \to \mathbb{C}\) must act as the identity on \(\mathbb{R}\): for \(x \in \mathbb{R}, \varphi(x) = x\); you do not need to prove this. Using this fact and part (a), show that the only field isomorphisms \(\varphi : \mathbb{C} \to \mathbb{C}\) are \(\varphi(z) = z\) and \(\varphi(z) = \overline{z}\).