MATH 140A: FOUNDATIONS OF REAL ANALYSIS I
SUMMARY OF KEY FACTS FOR EXAM 1

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Throughout, reference numbers (like Definition 1.2 and Theorem1.17) refer to the course lecture notes, also available on the website. References to homework exercises are formatted as $\langle$assignment$\rangle.\langle$problem$\rangle$.

Properties of Fields, Especially $\mathbb{R}$

(1) Definition of an ordered set (Definition 1.1).
(2) Definition of $\sup =$ least upper bound and $\inf =$ greatest lower bound.
(3) Completeness = least upper bound property (Definition 1.15). The rational numbers $\mathbb{Q}$ are not complete.
(4) Fields (Definition 1.6) and ordered fields (Definition 1.10).
(5) Complete fields are Archimedean, and contain $\mathbb{Q}$ densely (Theorem 1.17).
(6) The least upper bound property is equivalent to the nested intervals property (Proposition 1.19 and Homework 2.3).
(7) There is a unique complete ordered field; we call it $\mathbb{R}$, the real numbers (Theorem 1.20).
(8) $\mathbb{R}$ contains nth roots of positive numbers, for any $n \in \mathbb{N}$ (Theorem 1.23).

Sequences and Limits

(1) Definition of limits (Definition 2.2), which are unique (Lemma 2.4).
(2) Bounded monotone sequences in $\mathbb{R}$ converge (Proposition 2.6).
(3) Definition of Cauchy sequence (Definition 2.7).
(4) Convergent sequences are Cauchy (Lemma 2.8).
(5) Cauchy sequences are bounded (Proposition 2.10).
(6) Subsequences (Definition 2.11) respect limits and Cauchy-ness (Proposition 2.13).
(7) Squeeze Theorem (Theorem 2.14).
(8) Cauchy completeness (Definition 2.15).

Some Useful Tools about $<$

(1) If $a, b \in \mathbb{R}$:
  (a) $a \leq b$ iff $a < b + \epsilon$ for all $\epsilon > 0$ iff $a \leq b + \epsilon$ for all $\epsilon > 0$.
  (b) $a = b$ iff $a \leq b$ and $b \leq a$.
  (c) $a = b$ iff $|b - a| \leq \epsilon$ for all $\epsilon > 0$ iff $|b - a| < \epsilon$ for all $\epsilon > 0$.

(2) If $\varnothing \neq A \subset \mathbb{R}$ is bounded above and $\alpha \in \mathbb{R}$, to show that $\alpha = \sup A$ it is necessary and sufficient to show two things: that α is an upper bound for A, and given any $x < \alpha$ there is some element $a \in A$ with $a > x$. Similarly, if A is bounded below and $\beta \in \mathbb{R}$, to show $\beta = \inf A$ it is necessary and sufficient to show two things: that β is a lower bound for A, and given any $y > \beta$ there is some element $b \in A$ with $b < y$.

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