Math 140A: "Winter" 2016
Final Practice Problems

Thursday, March 17	3:00pm – 6:00pm	PETER 102

This is a three-hour exam. You may bring two double-sided sheet of notes. No calculators or other electronic devices, or other written materials are allowed during the exam. Point values (totaling 50) are indicated for each problem. Please do all your work in a **Blue Book** from the UC San Diego bookstore; make sure to write your name and student number clearly on each Blue Book you turn in. **Have fun!**

1. (10 points) Let $K \subset \mathbb{R}$ be a compact set. Prove from the definition of compactness (not using the Heine-Borel theorem) that $\inf K$ and $\sup K$ exist in $\mathbb{R}$.

2. (10 points) Let (x_n) be a sequence in a metric space X, and suppose (x_n) has *no* convergent subsequences. Prove that the set $\{x_1, x_2, x_3, \ldots\}$ is closed in X.

3. (10 points) Let $f(x) = \sqrt{x}$ defined on $[0, \infty)$. Prove that f is continuous on its domain. Is it uniformly continuous on $[0, 1]$? [*Hint*: You may wish to treat continuity at 0 separately from continuity at any other point $x_0 > 0$. The factorization $x - x_0 = (\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})$ is useful here.]

4. (10 points) Let $A \subseteq \mathbb{R}$ be open, and let $f\colon A \to [0, \infty)$ denote the function $f(x) = x^4$. Show that f is uniformly continuous if and only if A is bounded.

5. Let $X = \{\text{bounded functions } f\colon [0,1] \to \mathbb{R}\}$, with metric defined by

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

(a) (5 points) Let (f_n) be a Cauchy sequence in X. Prove that, for each $x \in X$, the sequence $(f_n(x))_{n=1}^{\infty}$ converges in $\mathbb{R}$.

(b) (5 points) Let $g(x) = \lim_{n\to\infty} f_n(x)$ from part **(a)**. Show that $g \in X$, and that $d(f_n, g) \to 0$. Conclude that every Cauchy sequence in X converges in X, i.e. X is Cauchy complete.