1. Exercise 9, p. 166 in Rudin.

2. Recall the Cantor Set $C$ (Example 5.34, p. 65 in the Lecture Notes). It is the intersection
   \[ C = \bigcap_n K_n \] of compact nested sets $K_{n+1} \subseteq K_n$ in the unit interval; $K_n$ is a disjoint
   union of $2^n$ intervals, each of length $\frac{1}{3^n}$. Define $g_n = \mathbb{1}_{K_n}$.
   (a) Show that $g_n$ is Riemann integrable on $[0, 1]$. Define $f_n : [0, 1] \to \mathbb{R}$ by
   \[ f_n(x) = \int_0^x g_n(t) \, dt. \]
   Prove that $f_n$ is continuous, monotone increasing, and maps $[0, 1]$ onto $[0, 1]$.
   (b) Show that $f_n(x) = f_{n+1}(x)$ for all $x \in K_n^c$, and more generally $|f_n(x) - f_{n+1}(x)| < \frac{1}{2^n}$ for all $x \in K_n$.
   (c) Conclude that there is a continuous function $f : [0, 1] \to [0, 1]$ so that $f_n \to_u f$. Further, show that $f$ is monotone increasing, $f$ maps $[0, 1]$ onto $[0, 1]$, and $f$ is differentiable on $C^c$ with $f'(x) = 0$ for $x \in C^c$.

[The function $f$ is sometimes called the Devil’s staircase. It is a continuous function that increases from 0 to 1 over the unit interval; but it is flat almost everywhere (i.e. everywhere except on the Cantor set, which has measure 0). Such functions are called singular. They show that the Fundamental Theorem of Calculus really requires the function to be differentiable everywhere; for here we have a function $f$ whose derivative $f'$ exists almost everywhere, but there is no way to extend $f'$ to be defined on $[0, 1]$ in such a way that $f(x) = \int_0^x f'(t) \, dt$, since that integral is 0 for all $x$, while $f(x) \neq 0$ for all $x > 0$.]

3. Let $E$ be a differentiable function on $\mathbb{R}$ which satisfies $E(0) = 1$ and $E'(x) = E(x)$ for all $x \in \mathbb{R}$. For any $y \in \mathbb{R}$, define $f_y : \mathbb{R} \to \mathbb{R}$ by
   \[ f_y(x) = E(x + y) - E(x)E(y). \]
   (a) Prove that $f_y$ is differentiable, $f'_y(x) = f_y(x)$ for all $x$, and $f_y(0) = 0$.
   (b) Prove that $f_y$ is analytic on $\mathbb{R}$. [Hint: Use Proposition 10.29 from the course notes.]
   (c) Conclude that $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$. 

Turn in the homework by 5:00pm in the homework box in the basement of AP&M. Late homework will not be accepted.
4. Let $K$ be a compact subset of $\mathbb{R}^d$, and let $f: K \times [a, b] \to \mathbb{R}$ be a continuous function. Prove that the function $F: K \to \mathbb{R}$ defined by

$$F(x) = \int_{a}^{b} f(x, t) \, d\alpha(t)$$

is continuous on $K$. [Hint: since $K \times [a, b]$ is compact, the function $f$ is uniformly continuous. The points $(x, t)$ and $(y, t)$ are close whenever $x$ and $y$ are close in $K$.]