Today: §2.2-2.3: Matrix Inverse

Next: §4.1: Vector Spaces & Subspaces

Today is the last day to drop a course without a 'W'

Homework:

MATLAB Assigment #3: Due Feb 9 (next Friday)

MyMathLab Homework #4: Due Feb

Back to gradeschool algebra:

Solve $ax = b$ for x.

Can we solve $A\underline{x} = \underline{b}$ the same way?

Eg. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{b}$ Note: $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$

What went wrong?

System $\underset{m\times n}{A}\underline{x} = \underline{b}$. $\quad \underset{n\times m}{C}$ s.t. $CA = I_n$

So $A\underline{x} = \underline{b} \Rightarrow \underbrace{CA\underline{x}}_{=\underline{x}} = C\underline{b}$

$\not\Leftarrow$

Now, suppose $AC = I_m$.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \, ; \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} =$$

Definition: Let A be an $n \times n$ matrix. It is called **invertible** if there is a matrix C satisfying

Then we denote

Solving Systems: $A\underline{x} = \underline{b}$

* Existence:

* Uniqueness:

Invertible matrices must be square.

$AA^{-1} = I$: * If $A\underline{x} = \underline{b}$ is solvable for any $\underline{b}$.

$A^{-1}A = I$: * If $A\underline{x} = \underline{b}$ has a unique solution whenever it is consistent,

E.g. $\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix} =$

$\begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} =$

E.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Relation to Linear Transformations

Matrix A $\rightsquigarrow$ Linear transformation $T(\underline{x}) = A\underline{x}$.

* $A\underline{x} = \underline{b}$ solvable for all $\underline{b}$ $\iff$ T is onto
* $A\underline{x} = \underline{b}$ has at most one solution for each $\underline{b}$ $\iff$ T is one-to-one.

$A^{-1}A = I$ $\rightsquigarrow$ T onto $\rightsquigarrow$ every row is pivotal.

$AA^{-1} = I$ $\rightsquigarrow$ T is one-to-one $\rightsquigarrow$ every column is pivotal.

For a square matrix, every row is pivotal iff every column is pivotal!

Theorem A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one if and only if it is onto.

For the same reason $\Big\}$ An $n \times n$ matrix A is invertible

iff A has n pivot positions

iff there exists $n \times n$ C with $CA = I$

iff there exists $n \times n$ D with $AD = I$

in both of these cases, $C = D = A^{-1}$:

$$C = CI = C(AD) = (CA)D = ID = D.$$

Moral: we only need to find a "one-sided" inverse.

Not all square matrices are invertible!
(Must have maximal pivots.)

Q: How do we find the inverse, if it exists?

To find the inverse: Take the (multi-) augmented matrix $[A \,\vdots\, I]$ and perform row operations until $A \rightsquigarrow \operatorname{rref}(A)$. If $\operatorname{rref}(A) = I$, then the right-hand-side is $I \rightsquigarrow A^{-1}$.

E.g. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

E.g. General 2×2 matrix

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

A

Theorem: A is invertible iff

and in this case

E.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$

E.g. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

Recap: A matrix A is invertible if there is a matrix C with $CA = I = AC$.

We denote $C = A^{-1}$ in this case.

* Only square matrices can be invertible
* A is invertible if and only if $\mathrm{rref}(A) = I$

i.e. every row of A is pivotal

i.e. every column of A is pivotal

i.e. the columns of A are linearly independent and span $\mathbb{R}^n$.

* To find A^{-1}, we use row reduction:

$$[\, A \mid I \,] \xrightarrow{\text{RREF}} [\, I \mid A^{-1} \,]$$