Today: § 4.1: Vector Spaces & Subspaces

& § 4.2: Null Spaces & Column Spaces

Next: § 4.3: Linear Independent Sets; Bases

Homework:

regrade requests:
see procedure on course webpage

MATLAB Assignment #3: Due Feb 9 (Friday)

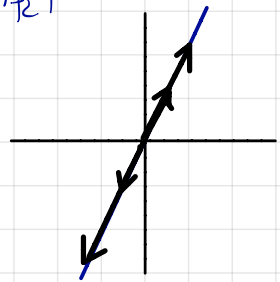
MyMathLab Homework #4: Due Feb 12 (Monday)

A **vector space** is a set V together with a special element 0 and equipped with two operations $+$ & scalar mult. satisfying all the usual rules.

Eg. $M_{2\times 3} = \{2\times 3 \text{ matrices}\}$ is a vector space.

$$\lambda \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix} = \begin{bmatrix} \lambda a_1 + a_2 & \lambda b_1 + b_2 & \lambda c_1 + c_2 \\ \lambda d_1 + d_2 & \lambda e_1 + e_2 & \lambda f_1 + f_2 \end{bmatrix}$$

Eg. The line $L = \left\{ t\begin{bmatrix} 1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$ is a vector space $s\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (s+t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in L$



Definition: If V is a vector space, and $W \subseteq V$ is nonempty, W a **subspace** of V if W is closed under addition and scalar multiplication (in V).

That is to say: W is closed under addition (in V)

i.e. if $\underline{u}, \underline{v} \in W \subseteq V$, then $\underline{u} + \underline{v} \in W$ (not just in V)

& W is closed under scalar multiplication (in V)

i.e. if $\lambda \in \mathbb{R}$ and $\underline{w} \in W$, then $\lambda \underline{w} \in W$ (not just in V)

& $\underline{0}$ (the zero vector in V) is in W.

E.g. $\{\underline{0}\}$ ← the trivial subspace.

Eg. $W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$. $\lambda \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ 0 \end{bmatrix} \in W$

$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$W \neq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Eg. $A = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}$ $\quad \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2 \end{bmatrix} \leftarrow \notin A$

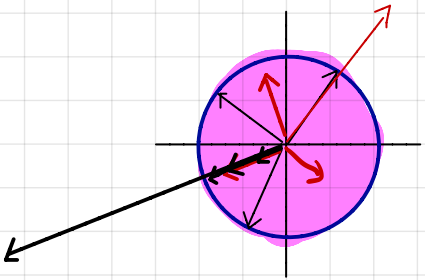
Eg. The set $\left\{ \begin{bmatrix} a & b & 0 \\ c & 0 & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \subseteq M_{2\times 3}$ $\quad$ = Bab is a subspace.

$= \text{Span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

$\underline{0} \in Bab$ ✓, $\begin{bmatrix} a_1 & b_1 & 0 \\ c_1 & 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & 0 \\ c_2 & 0 & d_2 \end{bmatrix} + \begin{bmatrix} a_1+a_2 & b_1+b_2 & 0 \\ c_1+c_2 & 0 & d_1+d_2 \end{bmatrix} \in Bab.$

Eg. $S^1 \subset \mathbb{R}^2$

Eg. $\mathbb{D} \subset \mathbb{R}^2$



not closed under scalar multiplication.

Eg. Let V be any vector space, and let $\underline{v}_1, \dots, \underline{v}_n$ be vectors in V. Then $\text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$ is a subspace of V.

Eg. $n=2$: $\underline{v}_1, \underline{v}_2$. $\overbrace{\text{Span}\{\underline{v}_1, \underline{v}_2\}}^{S} = \{x_1\underline{v}_1 + x_2\underline{v}_2 : x_1, x_2 \in \mathbb{R}\}$

* take $x_1, x_2 = 0$, $\underline{0} \in S$.
* $(x_1\underline{v}_1 + x_2\underline{v}_2) + (y_1\underline{v}_1 + y_2\underline{v}_2) = (x_1+y_1)\underline{v}_1 + (x_2+y_2)\underline{v}_2 \in S$ ✓
* S is closed under scalar mult.

Eg. $\left\{ \begin{bmatrix} a-2b \\ 2a+b \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ $\therefore$ it is a subspace.

$$\begin{bmatrix} a-2b \\ 2a+b \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

§4.2: Null Space & Column Space

Given any matrix $A \in M_{m\times n}$ there are two important subspaces: $A = [\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_n]$

The **null space** of A, $\mathrm{Nul}(A) = \{\underline{x} : A\underline{x} = \underline{0}\}$

= the sol'n set of the homog system whose coeff. matrix is A

The **column space** of A, $\mathrm{Col}(A) = \{\underbrace{A\underline{x}}_{m\times n\ n\times 1} : \underline{x} \in \mathbb{R}^n\}$

$= \{x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n\} = \mathrm{span}\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$.

We will see shortly that they are subspaces – but of what?

$\mathrm{Nul}(A) \subseteq \mathbb{R}^n \qquad \mathrm{Col}(A) \subseteq \mathbb{R}^m$

Theorem: For $A \in M_{m\times n}$, $Nul(A)$ is a subspace of $\mathbb{R}^n$ ✓
& $Col(A)$ is a subspace of $\mathbb{R}^m$ ✓

Pf. $Col(A) = span\{\text{columns of } A\}$ is a subspace. ✓

$Nul(A) = \{\underline{x} : A\underline{x} = \underline{0}\}$

* Is $\underline{0} \in Nul(A)$? I.e. is $A\underline{0} = \underline{0}$? Yes. ✓
* If $\underline{x}, \underline{y} \in Nul(A)$, is $\underline{x}+\underline{y} \in Nul(A)$?

I.e. $\underline{0} \overset{?}{=} A(\underline{x}+\underline{y})$

$$A(\underline{x}+\underline{y}) = \underset{\substack{\| \\ \underline{0}}}{A\underline{x}} + \underset{\substack{\| \\ \underline{0}}}{A\underline{y}} = \underline{0}+\underline{0} = \underline{0} \quad ✓$$

$\underline{x} \in Nul(A)$ $\qquad$ $y \in Nul(A)$

* Is $\lambda\underline{x} \in Nul(A)$?

$A(\lambda\underline{x}) = \lambda A\underline{x} = \lambda\underline{0} = \underline{0}$. ✓

E.g. $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s+t-3r \\ s \\ -2t+2r \\ t \\ r \end{bmatrix}$$

$$\mathrm{Nul}(A) = \{\underline{x} : A\underline{x} = \underline{0}\}$$

$$= s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$