Today: § 4.1: Vector Spaces & Subspaces
& § 4.2: Null Spaces & Column Spaces

Next: § 4.3: Linear Independent Sets; Bases

Homework: regrade requests:
see procedure on course webpage

MATLAB Assignment #3: Due Feb 9 (Friday)

MyMathLab Homework #4: Due Feb 12 (Monday)
A vector space is a set $V$ together with a special element $0$ and equipped with two operations $+$ & scalar mult. satisfying all the usual rules.

E.g. $M_{2\times 3} = \{ \text{2\times3 matrices} \}$ is a vector space.

$\lambda \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix} = \begin{bmatrix} \lambda a_1 + a_2 & \lambda b_1 + b_2 & \lambda c_1 + c_2 \\ \lambda d_1 + d_2 & \lambda e_1 + e_2 & \lambda f_1 + f_2 \end{bmatrix}$

E.g. The line $L = \{ t[1'] : t \in \mathbb{R} \}$ is a vector space $s[1'] + t[1'] = (s+t)[1'] \in L$
Definition: If $V$ is a vector space, and $W \subseteq V$ is nonempty, $W$ a subspace of $V$ if $W$ is closed under addition and scalar multiplication (in $V$).

That is to say: $W$ is closed under addition (in $V$) i.e. if $u, v \in W \subseteq V$, then $u + v \in W$ (not just in $W$) & $W$ is closed under scalar multiplication (in $V$) i.e. if $\lambda \in \mathbb{R}$ and $w \in W$, then $\lambda w \in W$ (not just in $W$) & $0$ (the zero vector in $V$) is in $W$.

E.g. $\{0\}$ ← the trivial subspace.
Eg. \( W = \{ [x] : x, y \in \mathbb{R} \} \) is a subspace of \( \mathbb{R}^3 \). \( [x] = [\frac{3x}{y}] \in W \), \( W + [0] = \text{span} \{ [0], [1] \} \).

Eg. \( A = \{ [x] : x, y \in \mathbb{R} \} \), \( [x_1] + [x_2] = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \notin A \).

Eg. The set \( \{ [a, b, c, d] : a, b, c, d \in \mathbb{R} \} \subset M_{2 \times 3} \), \( \text{Span} \{ [1, 0, 0], [0, 1, 0], [0, 0, 1] \} \).

\( \exists \in \text{Bob} \), \( \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ c_1 & 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ c_2 & 0 & d_2 \end{bmatrix} + \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & c_1 + c_2 & 0 \\ c_1 + c_2 & 0 & d_1 + d_2 \end{bmatrix} \in \text{Bob} \).

Eg. \( S^1 \subset \mathbb{R}^2 \).

Eg. \( D \subset \mathbb{R}^2 \).

not closed under scalar multiplication.
Eg. Let $V$ be any vector space, and let $v_1, \ldots, v_n$ be vectors in $V$. Then $\text{span}\{v_1, \ldots, v_n\}$ is a subspace of $V$.

Eg. $n=2 : v_1, v_2$. $\text{span}\{v_1, v_2\} = \{x_1v_1 + x_2v_2 : x_1, x_2 \in \mathbb{R}\}$

* take $x_1, x_2 = 0$, $0 \in S$, 
* $(x_1v_1 + x_2v_2) + (y_1v_1 + y_2v_2) = (x_1 + y_1)v_1 + (x_2 + y_2)v_2$ 
* $S$ is closed under scalar mult. 

Eg. $\left\{\begin{bmatrix} a-2b \\ 2a+b \\ b \end{bmatrix} : a, b \in \mathbb{R}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a subspace.
§4.2: Null Space & Column Space

Given any matrix \( A \in \mathbb{M}_{mxn} \), there are two important subspaces: \([a_1, a_2, \ldots, a_n]\)

The null space of \( A \), \( \text{Nul}(A) = \{ x : A x = 0 \} \)

The column space of \( A \), \( \text{Col}(A) = \{ A x : x \in \mathbb{R}^n \} \)

\( \{ x a_1 + x_2 a_2 + \ldots + x_n a_n \} = \text{span} \{ a_1, a_2, \ldots, a_n \} \).

We will see shortly that they are subspaces - but of what?

\( \text{Nul}(A) \subseteq \mathbb{R}^n \quad \text{Col}(A) \subseteq \mathbb{R}^m \)
**Theorem:** For $A \in M_{m \times n}$, $\text{Null}(A)$ is a subspace of $\mathbb{R}^m$.

Prove: $\text{Col}(A) = \text{span}\{\text{columns of } A\}$ is a subspace. 

$\text{Null}(A) = \{x : Ax = 0\}$

* Is $0 \in \text{Null}(A)$? I.e., is $A0 = 0$? Yes. ✓

* If $x, y \in \text{Null}(A)$, is $x + y \in \text{Null}(A)$?

I.e., $0 = A(x + y)$

$A(x + y) = Ax + Ay = 0 + 0 = 0$ ✓

* Is $\lambda x \in \text{Null}(A)$?

$I.e., \lambda = A(\lambda x) = \lambda Ax = \lambda 0 = 0$. ✓
E.g. \( A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \)

\[ \text{null}(A) = \{ x : Ax = 0 \} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\} \]