Today: § 4.2: Null Space & Column Space

& § 4.3: Linearly Independence & Bases

Next: § 4.5-4.6: Dimension & Rank

Homework:

MATLAB Assigment #3: Due TONIGHT.

MyMathLab Homework #4: Due Monday (Feb 12)

E.g. $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$ $\quad \underline{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \\ 4 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

$\operatorname{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Is $\underline{u} \in \operatorname{Nul}(A)$?

Is $\underline{v} \in \operatorname{Col}(A)$?

Question: Is the solution set $\{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b}\} = W$ a subspace, if $\underline{b} \neq \underline{0}$?

Answer:

Is $\underline{0} \in W$?

Is W closed under scalar multiplication?

Is W closed under addition?

§4.3 Remember **linear dependence**:

Vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ in some vector space V

are linearly dependent if

$\iff$

The vectors are **linearly independent** if

they are not linearly dependent.

$\iff$

E.g. Polynomials $(x+1)^2,\ x^2,\ x+\frac{1}{2}$:

Definition: Let V be a vector space, and let $H \subseteq V$ be a subspace.

A collection $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_d\}$ of vectors in H is called a **basis** for H if:

E.g. The "standard basis vectors" $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\dots$, $\underline{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

are a basis for $\mathbb{R}^n$.

E.g. Is $\left\{ \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Theorem: Any basis of $\mathbb{R}^n$ consists of exactly n column vectors. The set $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ is a basis iff the matrix $[\underline{v}_1 \ \underline{v}_2 \ \cdots \ \underline{v}_n]$ is invertible.

Eg. Let $W \subseteq \mathbb{R}^3$ be the set of all vectors of the form $\begin{bmatrix} a+b \\ a-b \\ 2a \end{bmatrix}$.

Then $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for W.

What about $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}$?

$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$?

Theorem: Let $\{\underline{v}_1, \dots, \underline{v}_n\}$ be vectors, and let $H = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$. If the vectors are linearly <u>dependent</u>, then there is at least one vector $\underline{v}_k$ such that

$$H = \text{span}\{\underline{v}_1, \dots, \underline{v}_{k-1}, \underline{v}_{k+1}, \dots, \underline{v}_n\}$$

I.e. H is spanned by the vectors with<u>out</u> $\underline{v}_k$.

Corollary: If $H = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$, then some collection of these vectors is a basis for H.

E.g. $H = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix} \right\}$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The column space $\mathrm{Col}(A)$ of a matrix A is (by definition) spanned by the columns of A.

But these may in general be linearly dependent: The non-pivotal columns are linear combinations of the pivotal columns; but the pivotal columns are linearly independent.

$\Longrightarrow$ <u>Theorem</u>: The pivotal columns of A form a basis for $\mathrm{Col}(A)$.