Today: § 4.2-4.3: Bases of Col(A), Nul(A)

& § 4.5-4.6: Dimension & Rank

Next: § 4.4 & 4.7: Change of Basis

Homework:

MyMathLab Homework #4: Due ~~TONIGHT~~ Tuesday by 11:59pm

MyMathLab Homework #5: Due Feb 20 by 11:59pm

A **basis** of a subspace H is a collection of vectors

$$\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_d\} \subset H$$

that (1) spans H, and

(2) is linearly independent.

Subspaces are frequently presented as the span of some collection of vectors; but those vectors may not be independent. How do we find a basis?

$$H = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Corollary: If $H = \text{span}\{\underline{v}_1, \ldots, \underline{v}_n\}$, then some collection of these vectors is a basis for H.

E.g. $H = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem: The pivotal columns of A form a basis for $\mathrm{Col}(A)$. (The coefficients expressing the non-pivotal columns in terms of this basis can be read off of $\mathrm{rref}(A)$.)

Eg. $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & -1 \\ 2 & 1 & -1 \end{bmatrix}$

Caution: The pivotal columns of A, **not** $\mathrm{rref}(A)$, form a basis for $\mathrm{Col}(A)$. Typically

$$\mathrm{Col}(A) \neq \mathrm{Col}(\mathrm{rref}(A)).$$

How about $\mathrm{Nul}(A)$? How do I find a basis?

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Theorem</u>: This procedure for identifying $\mathrm{Nul}(A)$ always produces a basis for it.

A given subspace always has zillions of bases. But there is one thing in common among all bases.

Theorem: Any two bases of a vector space have the same number of vectors.

Pf.

Definition: If V is a vector space, its **dimension** is the number of vectors in any basis.

E.g. $V = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$

We saw that the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ has 2 pivotal columns. $V = \mathrm{Col}(A)$, so

$$\dim(V) =$$

E.g. $\mathbb{P}_3 = \{\text{polynomials of degree} \leq 3\}$

E.g. $\mathbb{P} = \{\text{polynomials}\}$

Theorem: If $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly independent, it can be extended to form a basis:

$$\{\underline{v}_1, \dots, \underline{v}_n, \underline{u}_1, \dots, \underline{u}_p\}.$$

If $\{\underline{w}_1, \dots, \underline{w}_n\}$ is a spanning set, it can be reduced to form a basis.

Corollary: If H is a subspace of V, then

$$\dim(H) \leq \dim(V).$$

Definition: Given a matrix A,

$$\operatorname{rank}(A) := \dim(\operatorname{Col}(A))$$
$$\operatorname{nullity}(A) := \dim(\operatorname{Nul}(A)).$$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank Theorem: If A is an $m \times n$ matrix,

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

$$(= \#\text{Columns of } A)$$