Today: § 4.6 Rank & § 4.4 Coordinates

Next: § 3.1 - 3.2: Determinants

Homework:

MyMathLab Homework #5: Due Tuesday

MATLAB Assigment #4: Due February 23

Midterm #2: 2 weeks from tonight.

The **dimension** of a vector space V is the number of vectors in any basis. (Makes sense, since all bases have the same number of vectors.)

E.g. $V = \left\{ \begin{bmatrix} a+b \\ a-b \\ 2a \end{bmatrix} : a, b \in \mathbb{R} \right\}$

Definition: Given a matrix A,

$$\operatorname{rank}(A) := \dim(\operatorname{Col}(A))$$

$$\operatorname{nullity}(A) := \dim(\operatorname{Nul}(A)).$$

$$A = \begin{bmatrix} 1 & -3 & 6 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(A)$$

Theorem: The pivotal columns of A form a basis for $Col(A)$.
The non-pivotal columns of A correspond to a basis for $Nul(A)$.

$\therefore$ $rank(A) =$

& $nullity(A) =$

$\therefore$ $rank(A) + nullity(A) =$ ____________

One more important subspace:

Definition: The **row space** of A, $Row(A)$, is the span of the rows of A.

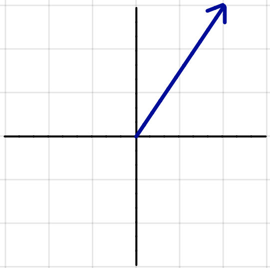
Eg. $A = \begin{bmatrix} 1 & 1 & 0 & -1 & 2 \\ 3 & 3 & 0 & -3 & 6 \\ 1 & 2 & 3 & 0 & 6 \end{bmatrix}$

$\mathrm{Row}(A) = \mathrm{span}\left\{ [1\ 1\ 0\ {-1}\ 2], [3\ 3\ 0\ {-3}\ 6], [1\ 2\ 3\ 0\ 6] \right\}$

Theorem: $\mathrm{Row}(A)$ $\qquad$ $\mathrm{Row}(\mathrm{rref}(A))$

$\dim(\mathrm{Row}(A))$

§4.4 What's so great about a basis, anyway?



<u>Theorem</u>: If $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ is a basis for V, then each vector $\underline{v} \in V$ has a <u>unique</u> expansion

$$\underline{v} = x_1 \underline{b}_1 + x_2 \underline{b}_2 + \cdots + x_n \underline{b}_n$$

for some <u>unique</u> scalars $x_1, \dots, x_n \in \mathbb{R}$.

This means, if V has a basis $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ (so $\dim(V) = n$) then we can identify V with $\mathbb{R}^n$ by identifying each vector $\underline{v}$ with its $\mathcal{B}$-coordinate vector

$$\underline{v} = x_1 \underline{b}_1 + x_2 \underline{b}_2 + \cdots + x_n \underline{b}_n \rightsquigarrow [\underline{v}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

Eg. If $\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, then its coordinate vector with respect to the standard basis $\mathcal{E} = \{\underline{e}_1, \underline{e}_2\}$ is just $[\underline{v}]_{\mathcal{E}} = \begin{bmatrix} a \\ b \end{bmatrix}$.

But what if we use the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$?

E.g. $\mathbb{P}_2$ = {polynomials of degree ≤ 2}

"Standard" basis $\mathcal{B} = \{1, x, x^2\}$

Then the polynomial $p = (x-1)^2 =$

Theorem: If $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ is a basis for V, then the function $T : V \to \mathbb{R}^n : T(\underline{v}) = [\underline{v}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Such a linear transformation is called an

E.g. $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\left\{ \qquad \qquad \right\} = \mathcal{B}$ basis for $\mathrm{Col}(A)$.

The point of isomorphisms is: they preserve all linear properties.

<u>Theorem</u>: Let $\{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\} = \mathcal{B}$ be a basis for V.

Then * $\{\underline{v}_1, \dots, \underline{v}_k\} \in V$ are linearly independent in V

iff $[\underline{v}_1]_{\mathcal{B}}, \dots, [\underline{v}_k]_{\mathcal{B}}$ are linearly independent in $\mathbb{R}^n$.

* " " ______ span ______ V

" " ______ span ______ $\mathbb{R}^n$.

Eg. Show that $\{1, x-1, (x-1)^2\}$ is a basis for $\mathbb{P}^2$.