Today: §3.3 Determinants & §5.1: Eigenvalues

Next: §5.2: Characteristic Polynomial

Reminders:
MATLAB Assignment #4: Due TONIGHT by 11:59pm
MyMathLab Homework #6: Due Tuesday by 11:59pm
Midterm #2: Next Wednesday, 8-10pm Room/seat assgn. & practice midterms posted on course webpage.
Properties of Determinants

* Defined by cofactor expansion (recursive expression, expand along any row or column)

* Behaves nicely under row ops: \( \det A = \pm (\text{product of pivots in row reduction to rref}(A)) \)

* \( \det A = 0 \) iff \( A \) is not invertible

* \( \det : M_n \rightarrow \mathbb{R} \) is multilinear in each row and column of the matrix, with the others held fixed.

* \( \det(AB) = \det(A)\det(B) \). \( \therefore \) \( \det(A^n) = (\det(A))^n \).

Two More Properties:

\( \det(A^{-1}) = \)

\( \det(A^T) = \)
What is the determinant? What does it mean?

Parallelograms:
Theorem: Given two vectors $a, b \in \mathbb{R}^2$, the area of the parallelogram they determine is

$$A(a, b) = |\det [a, b]|$$

$$= |\det \begin{bmatrix} a^T \\ b^T \end{bmatrix}|$$

What does the sign of $\det [a, b]$ mean?

What about $n \times n$ determinants with $n>2$?
Parallelepipeds
A linear transformation $T : V \to V$ tends to move vectors around.

E.g. $T([x]) = [\frac{3x - 2y}{x}]$

$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) =$

$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) =$

E.g. If $A$ is a stochastic matrix (row sums = 1) then
**Definition:** Let $T: V \rightarrow V$ be a linear transformation. If there is a non-zero $v \in V$ such that $T(v) = \lambda v$ for some scalar $\lambda \in \mathbb{R}$, we call $v$ an eigenvector of $T$. The scalar $\lambda$ is the corresponding eigenvalue. (We similarly talk about eigenvectors/eigenvalues of square matrices.)

**E.g.** Show that 7 is an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.
Theorem: For any \( \lambda \in \mathbb{R} \), the set of vectors \( v \) for which \( T(v) = \lambda v \) is a subspace of \( V \), called the eigenspace of \( \lambda \).

(We only call \( \lambda \) an eigenvalue if its eigenspace is \( \neq \{0\} \).)
Eg. The rotation matrix \[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \quad (ccw 90^\circ)
\] has no eigenvalues.
E.g. Let \( A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} \). 2 is an eigenvalue for \( A \); find a basis for the eigenspace.
* Given a (potential) eigenvalue, finding the corresponding eigenspace is routine.

* Finding the eigenvalues is typically hard. (Row reduction changes the eigenvalues.)

One case where eigenvalues can be read off:

**Theorem:** If A is triangular, its eigenvalues are its diagonal entries.
E.g. \[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \]

E.g. \[ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]