Today §6.1/6.7: Inner Products

Next: §6.2 : Orthogonality

Reminders:

MyMathLab Homework #7: Due THURSDAY by 11:59pm

MATLAB Homework #5: Due FRIDAY by 11:59pm

Final Exam is on Saturday, March 17 11:30am - 2:30pm

@ the BEGINNING of Exam week.

An $n\times n$ matrix A is diagonalizable if $A = PDP^{-1}$ ← invertible, diagonal

This is equivalent to $\mathbb{R}^n$ having a basis of eigenvectors for A.

This happens, e.g., when A has n distinct real eigenvalues. But it can fail.

↳ Eigenvalues: roots of $p_A(\lambda) = \det(A-\lambda I)$ ← degree n polynomial

must have all real roots for A to be diagonalizable.

↳ The algebraic multiplicity of λ is the degree of this root in p_A

The geometric multiplicity of λ is $\dim \mathrm{Nul}(A-\lambda I)$.

Always have geom. mult $(\lambda) \leq$ alg. mult. (λ) can be $\lneq$

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 5 \\ 0 & 0 & 5 \end{bmatrix} \quad p_A(\lambda) = p_B(\lambda) = (5-\lambda)^2(4-\lambda).$$

$\dim \mathrm{Nul}(A-5I) = 2$, $\dim \mathrm{Nul}(B-5I) = 1$.

Theorem: If A has a repeated eigenvalue λ, then
$\dim \mathrm{Nul}(A-\lambda I) \leq$ multiplicity of λ.
The matrix is diagonalizable if and only if p_A has only real roots, and $\dim \mathrm{Nul}(A-\lambda I) =$ multiplicity of λ for each λ.
In this case, any bases for the eigenspaces together form a basis for $\mathbb{R}^n$.

So, just which matrices are diagonalizable?
That's a really tough question to answer. A partial answer will be given at the end of the course. To get there...

§6.1 **Definition:** The dot product or inner product on $\mathbb{R}^n$ is defined by

$$\underline{u} \cdot \underline{v} = \langle \underline{u}, \underline{v} \rangle = \underline{u}^T \underline{v}$$

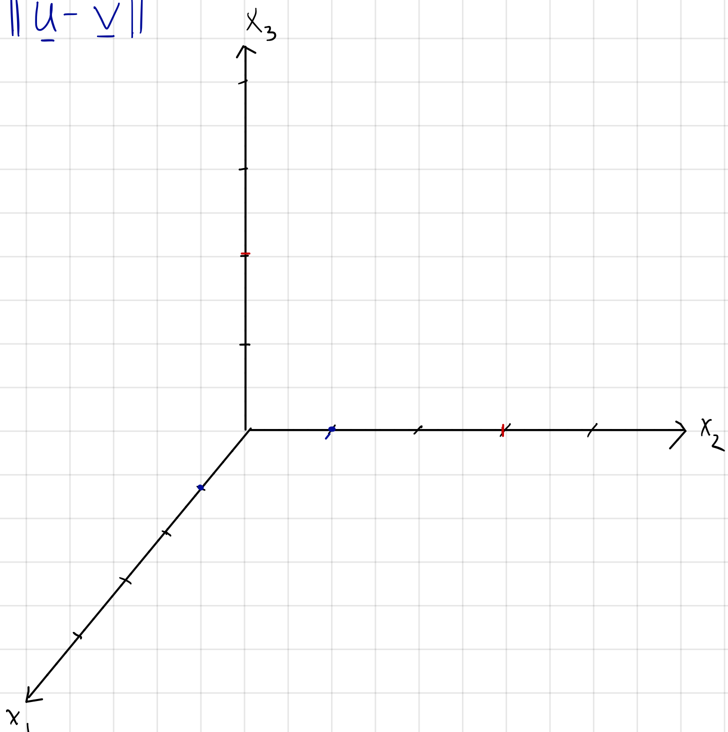
Properties:

The length of a vector is defined to be

$$\| \underline{v} \| = \sqrt{\underline{v} \cdot \underline{v}}$$

More generally, the distance between two vectors $\underline{u}, \underline{v}$ is

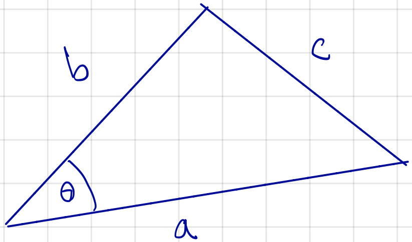
$$\text{dist}(\underline{u}, \underline{v}) = \| \underline{u} - \underline{v} \|$$



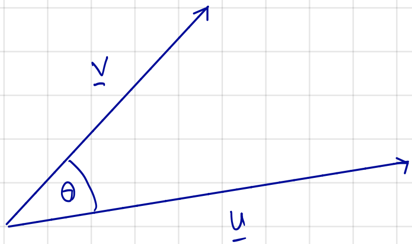
E.g. $\underline{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

How does $\|\underline{u} - \underline{v}\|$ relate to $\|\underline{u}\|$, $\|\underline{v}\|$?

Law of cosines (in 2 dimensions):



$$a^2 + b^2 = c^2 + 2ab\cos\theta$$



Cauchy-Schwarz Inequality

$$|\underline{u} \cdot \underline{v}| \leq \|\underline{u}\| \, \|\underline{v}\| .$$