Today §6.2 : Orthogonality

Next: §6.3 : Orthogonal Projections

Reminders:

MyMathLab Homework #7: Due THURSDAY by 11:59pm

MATLAB Homework #5: Due FRIDAY by 11:59pm

MATLAB QUIZ: Tuesday, March 13 @ your usual section time

Conflict quiz times on Monday.

The inner product on $\mathbb{R}^n$ encodes both lengths and angles:

$$\|\underline{v}\| = \sqrt{\underline{v}\cdot\underline{v}}\,, \qquad \underline{v}\cdot\underline{w} = \|\underline{v}\|\,\|\underline{w}\|\cos\theta$$

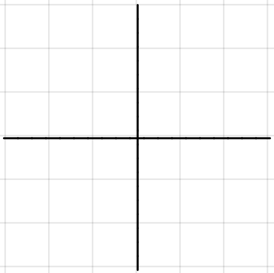
← angle between $\underline{v}$ and $\underline{w}$.

In particular, with $\theta = 0$, we see

two vectors $\underline{v}, \underline{w}$ are <u>orthogonal</u> iff $\underline{v}\cdot\underline{w} = 0$.

Pythagorean Theorem: $\underline{u} \perp \underline{v}$ iff $\|\underline{u}-\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$

<u>Definition</u>: If $V \subseteq \mathbb{R}^n$, the orthogonal complement of V, denoted $V^\perp$, is defined to be



Theorem: For any $m \times n$ matrix A,

$$(\text{Row } A)^{\perp} = \text{Nul } A$$

$$(\text{Col } A)^{\perp} = \text{Nul } A^{T}$$

Orthogonal Sets of Vectors.

Eg. The standard basis vectors are all orthogonal.

Eg. $\underline{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\underline{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$ are all orthogonal.

Theorem: If $\{\underline{u}_1, \dots, \underline{u}_p\}$ are orthogonal, they are linearly independent.
(all $\neq \underline{0}$)

Definition: Let $V \subseteq \mathbb{R}^n$ be a subspace. A basis for V is called an **orthogonal basis** if all the basis vectors are orthogonal. It is called an **orthonormal basis** if, in addition, each basis vector has length 1.

Eg. The standard basis

Eg. $\left\{ \underline{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \underline{u}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \underline{u}_3 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$.

Theorem: If $\mathcal{B} = \{\underline{u}_1, \dots, \underline{u}_p\}$ is an orthogonal basis for a subspace V, then the coefficients of any vector $\underline{v} \in V$ in the basis $\mathcal{B}$ are

$$[\underline{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}, \qquad c_j = \frac{\underline{v} \cdot \underline{u}_j}{\|\underline{u}_j\|^2}$$

Orthogonality & Matrices

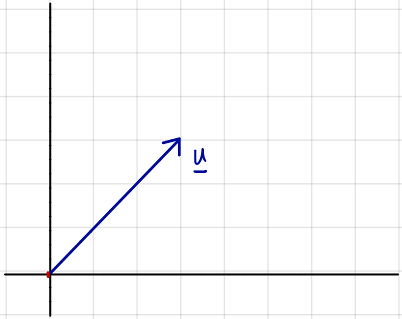
Given an $m \times n$ matrix A, the matrix $A^T A$ encodes the dot products of the columns of A

If the columns are orthonormal, then

Definition: If the columns of an $n \times n$ matrix U form an orthonormal basis for $\mathbb{R}^n$, we call U an orthogonal matrix. This is equivalent to

Better yet: thinking of such a U as a linear transformation,

Orthogonal Projections



E.g.

$$\underline{u} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

What's really going on:

If $V \subseteq \mathbb{R}^n$ is a subspace, it has an orthogonal complement $V^\perp \subseteq \mathbb{R}^n$. The two are complementary: $\dim V + \dim V^\perp = n$.

They are also linearly independent.

Thus, any vector $\underline{y} \in \mathbb{R}^n$ has a unique decomposition: