Today §6.3: Orthogonal Projections

Next: §6.4: Orthogonalization

Reminders: Please fill out your CAPEs.

MyMathLab Homework #8: Due March 15 by 11:59pm

MATLAB Homework #5: Due TONIGHT by 11:59pm

MATLAB QUIZ: Tuesday, March 13 @ your usual section time

Conflict quiz times on Monday.

A collection $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p$ of vectors in $\mathbb{R}^n$ is orthogonal if $\underline{u}_i \cdot \underline{u}_j = 0$

They are orthonormal if, in addition, $\|\underline{u}_i\| = 1$ for all i, j.

Set

$$U = \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_p \end{bmatrix} \qquad (n \times p \text{ matrix})$$

Then the (i,j)-entry of $U^T U$ is $\langle \underline{u}_i, \underline{u}_j \rangle$. Therefore

$\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p\}$ are orthonormall if and only if

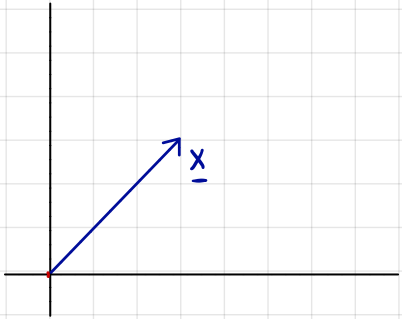
$$U^T U =$$

Note: if $n \neq p$, this does not mean that $UU^T =$

Definition: If the columns of an $n \times n$ matrix U form an orthonormal basis for $\mathbb{R}^n$, we call U an **orthogonal matrix**. This is equivalent to

Better yet: thinking of such a U as a linear transformation,

Orthogonal Projections



E.g.

$$\underline{u} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

What's really going on:

If $V \subseteq \mathbb{R}^n$ is a subspace, it has an orthogonal complement $V^\perp \subseteq \mathbb{R}^n$. The two are complementary: $\dim V + \dim V^\perp = n$.

Vectors in V are linearly independent from vectors in $V^\perp$.

Thus, any vector $\underline{y} \in \mathbb{R}^n$ has a unique decomposition:

How do we find Proj_V for a given subspace V ?

We already saw that if $V = \mathrm{span}\{\underline{u}\}$ (1-dimensional)

$$\mathrm{Proj}_V(\underline{y}) = \frac{\underline{y}\cdot\underline{u}}{\underline{u}\cdot\underline{u}}\,\underline{u}$$

Theorem: Let $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p\}$ be an orthogonal basis for V. Then

$$\mathrm{Proj}_V(\underline{y}) = \frac{\underline{y}\cdot\underline{u}_1}{\|\underline{u}_1\|^2}\,\underline{u}_1 + \frac{\underline{y}\cdot\underline{u}_2}{\|\underline{u}_2\|^2}\,\underline{u}_2 + \cdots + \frac{\underline{y}\cdot\underline{u}_p}{\|\underline{u}_p\|^2}\,\underline{u}_p$$

This allows us to compute the matrix of Proj_V.

Start with an orthonormal basis for V; then

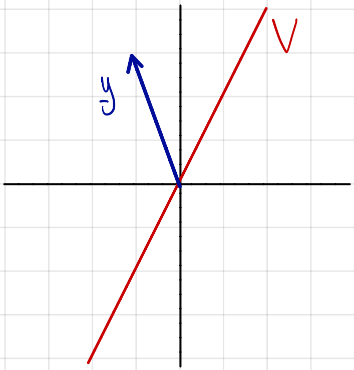
$$\text{Proj}_V(\underline{y}) = (\underline{y}\cdot\underline{u}_1)\underline{u}_1 + (\underline{y}\cdot\underline{u}_2)\underline{u}_2 + \cdots + (\underline{y}\cdot\underline{u}_p)\underline{u}_p$$

Theorem: Let $V \subseteq \mathbb{R}^n$ be a subspace, and fix an orthonormal basis $\{\underline{u}_1, \ldots, \underline{u}_p\}$ for V. Let $U = [\underline{u}_1\ \underline{u}_2 \cdots \underline{u}_p]$.

$\therefore\ \text{Proj}_V(\underline{y}) =$

Eg Compute the matrix of the orthogonal projection in $\mathbb{R}^3$ onto the subspace $\text{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\right\}$

What is an orthogonal projection, really?



Theorem: $\mathrm{Proj}_V(\underline{y})$ is the point in V that is closest to $\underline{y}$.

I.e. it is the best approximation of $\underline{y}$ in V.

E.g. $\left\{ \underline{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \underline{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for V.

Find the closest point in V to $\underline{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.