Today §6.4: Orthogonalization

Next: §7.1: The Spectral Theorem

Reminders: Please fill out your CAPEs.

MyMathLab Homework #8: Due THURSDAY by 11:59pm

MATLAB QUIZ: TOMORROW OR TODAY!

FINAL EXAM: This Saturday, 11:30am – 2:30pm

Seating / Room Assignment on Triton Ed

Given an orthonormal basis $\{\hat{\underline{u}}_1, \hat{\underline{u}}_2, \dots, \hat{\underline{u}}_p\}$ for a subspace $V \subseteq \mathbb{R}^n$, the orthogonal projection of a vector $\underline{x} \in \mathbb{R}^n$ into V is

$$\text{Proj}_V(\underline{x}) = (\underline{x}\cdot\hat{\underline{u}}_1)\hat{\underline{u}}_1 + (\underline{x}\cdot\hat{\underline{u}}_2)\hat{\underline{u}}_2 + \cdots + (\underline{x}\cdot\hat{\underline{u}}_p)\hat{\underline{u}}_p$$

$$= UU^T\underline{x} \quad \text{where } U = [\hat{\underline{u}}_1 \ \hat{\underline{u}}_2 \cdots \hat{\underline{u}}_p]$$

This is the unique vector $\underline{y} \in V$ such that $\underline{y} - \underline{x} \in V^\perp$.

Geometrically, it is the closest point in V to $\underline{x}$.

E.g. $\left\{ \underline{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \underline{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for V.

Find the closest point in V to $\underline{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

6.4/ What if you aren't given an orthogonal basis? Can you even be sure that one exists?

Eg. $V = \text{Nul}\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & 0 & 3 \end{bmatrix}$

The Gram-Schmidt Orthogonalization Process

Given a basis $\mathcal{B} = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p\}$ of a subspace V, we can find a new basis $\mathcal{O} = \{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_p\}$ with the properties:

(1) $\mathcal{O}$ is orthogonal

(2) $\mathrm{span}\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\} = \mathrm{span}\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k\}$ for $1 \leq k \leq p$.

The Gram-Schmidt Process takes a basis $\mathcal{B} = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ for a subspace V and turns it into an orthogonal basis $\mathcal{O} = \{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n\}$

You can then produce an orthonormal basis $\hat{\mathcal{O}} = \{\hat{\underline{u}}_1, \hat{\underline{u}}_2, \dots, \hat{\underline{u}}_n\}$ by normalizing.

Eg. Find an orthonormal basis for Col A where $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ -2 & 0 & -2 & 0 \\ 2 & 4 & 6 & 9 \end{bmatrix}$

Question: How can you recover the original vectors $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \dots\}$ from the new o.n. basis $\{\hat{\underline{u}}_1, \hat{\underline{u}}_2, \hat{\underline{u}}_3, \dots\}$?

Theorem: If A is an $m \times n$ matrix with linearly independent columns, then A has a factorization $A = QR$

$Q^T Q = I \rightarrow Q$

Eg.
$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 2 & 4 & 9 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/3 & 0 & -4/\sqrt{3} \\ -2/3 & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}}_{Q} \begin{bmatrix} & & \\ 0 & & \\ 0 & 0 & \end{bmatrix}$$

What is this good for?