Today § 7.1: The Spectral Theorem

Next: Review

<u>Reminders</u>: Please fill out your CAPEs.

MyMathLab Homework #8: Due TOMORROW by 11:59pm

FINAL EXAM: This Saturday, 11:30am–2:30pm
GH 242, PETER 108, YORK 2722
Seating / Room Assignment on Triton Ed

Given a collection $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ of linearly independent vectors in $\mathbb{R}^n$, the **Gram-Schmidt orthogonalization process** produces a new collection $\{\hat{\underline{u}}_1, \hat{\underline{u}}_2, \dots, \hat{\underline{u}}_p\}$ of orthonormal vectors, with the same span as the $\underline{v}_j$'s.

$$\underline{u}_1 = \underline{v}_1 \qquad \hat{\underline{u}}_1 = \underline{u}_1 / \|\underline{u}_1\|$$

$$\underline{u}_2 = \underline{v}_2 - \frac{\underline{v}_2 \cdot \underline{u}_1}{\|\underline{u}_1\|^2} \underline{u}_1 \qquad \hat{\underline{u}}_2 = \underline{u}_2 / \|\underline{u}_2\|$$

$$\underline{u}_3 = \underline{v}_3 - \frac{\underline{v}_3 \cdot \underline{u}_1}{\|\underline{u}_1\|^2} \underline{u}_1 - \frac{\underline{v}_3 \cdot \underline{u}_2}{\|\underline{u}_2\|^2} \underline{u}_2 \qquad \hat{\underline{u}}_3 = \underline{u}_3 / \|\underline{u}_3\|$$

$$\vdots$$

The triangular pattern here can be summarized by noting this means

$$\underbrace{\begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_p \end{bmatrix}}_{A\ (n \times p)} = \underbrace{\begin{bmatrix} \hat{\underline{u}}_1 & \hat{\underline{u}}_2 & \cdots & \hat{\underline{u}}_p \end{bmatrix}}_{Q\ (n \times p)} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ 0 & r_{22} & \cdots & r_{2p} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & r_{pp} \end{bmatrix}}_{R\ (p \times p)}$$

these come up in the G-S process

Eg. $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ -2 & -1 \end{bmatrix}$ Produce an o.n.b. of $\text{span}\{\underline{v}_1, \underline{v}_2\} = \text{Col}(A)$

$\quad\quad\quad\quad \underline{v}_1 \quad \underline{v}_2$

$\underline{u}_1 = \underline{v}_1 \qquad \|\underline{u}_1\|^2 = 2^2 + 1^2 + (-2)^2 = 9 \quad \therefore \|\underline{u}_1\| = 3.$

$\underline{v}_3 \cdot \underline{u}_1 = \underline{v}_2 \cdot \underline{v}_1 = 3.$

$\hat{\underline{u}}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}.$

$$\underline{u}_2 = \underline{v}_2 - \frac{\underline{v}_2 \cdot \underline{u}_1}{\|\underline{u}_1\|^2}\underline{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{3}{9}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$(\underline{v}_2 - (\underline{v}_2 \cdot \hat{\underline{u}}_1)\hat{\underline{u}}_1)$$

$$\|\underline{u}_2\|^2 = \left(-\tfrac{2}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$$

$$\begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix} = \hat{\underline{u}}_2$$

$$\underline{A = QR} \qquad Q = \begin{bmatrix} \hat{\underline{u}}_1 & \hat{\underline{u}}_2 \end{bmatrix} = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ -2/3 & -1/3 \end{bmatrix}$$

$\leftarrow$ orthonormal columns.

$\therefore Q^T Q = I$

$\therefore Q^T A = Q^T Q R$

$$= IR = R = \begin{bmatrix} 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$

Note: $Q^TQ = I$ does not mean $Q^{-1} = Q^T$! (Only if Q is square; in general QQ^T = orthogonal projection onto $Col(Q)$.)

Why should you care about QR-factorization?

Take the square matrix case.

$A = QR \rightsquigarrow A_1 = RQ = Q^{-1}AQ \sim A$ similar $\therefore$ they have the same eigenvalues

$Q^TA = R$

$Q^{-1}A =$

$A_1 = Q_1R_1$; $A_2 = R_1Q_1 = Q_1^{-1}A_1Q_1 \sim A_1 \sim A$

$\vdots$

$$A, A_1, A_2, A_3, \ldots, A_{100}, \ldots$$

<u>Thm</u>: $A_n \xrightarrow[n\to\infty]{}$ (rapidly) to an upper-Δ matrix.

7.1/ Recall that $A \in M_{n\times n}$ is **diagonalizable** if there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A. ($\Leftrightarrow$ $A = PDP^{-1}$.)

What kinds of matrices have an **orthonormal** basis of eigenvectors?

$$A = PDP^{-1} = PDP^T$$

eigenvector matrix; if they are o.n., $P^{-1} = P^T$.

$$\therefore\ A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A \quad \text{"symmetric"}$$

<u>Conclusion</u>: if A is orthogonally diagonalizable, then $A = A^T$.

What about the converse? $A\underline{u} = \lambda \underline{u}$, $A\underline{v} = \mu \underline{v}$ $(\lambda \neq \mu)$

$$(\lambda - \mu)\underline{u}\cdot\underline{v} = \lambda \underline{u}\cdot\underline{v} - \mu \underline{u}\cdot\underline{v} = (\lambda\underline{u})\cdot\underline{v} - \underline{u}\cdot(\mu\underline{v}) = (A\underline{u})\cdot\underline{v} - \underline{u}\cdot(A\underline{v})$$

$$(A\underline{u})^T\underline{v} - \underline{u}^T A\underline{v} = \underline{u}^T A^T \underline{v} - \underline{u}^T A \underline{v} = 0.$$

$$\boxed{(A\underline{u})^T = \underline{u}^T A^T}$$

$\therefore\ \underline{u}\cdot\underline{v} = 0$ i.e. $\underline{u} \perp \underline{v}$.

Suppose A is symmetric, and happens to be diagonalizable.

$\therefore$ The eigenspaces of A span all of $\mathbb{R}^n$.

↳ we just saw that eigenvectors for distinct eigenvalues are $\perp$.

So eigenspaces for distinct eigenvalues are orthogonal.

↳ if an eigenvalue has geometric multiplicity > 1, can take any basis of this eigenspace and produce an o.n.b. ( )

<u>Conclusion</u>: There is an o.n.b. of eigenvectors of A.

I.e. A is orthogonally diagonalizable.

The Spectral Theorem

Every symmetric matrix is orthogonally diagonalizable.

This gives the <u>spectral decomposition</u> of a symmetric matrix:

$$(A^T =)\ A = PDP^T = \begin{bmatrix} \hat{\underline{u}}_1 & \hat{\underline{u}}_2 & \cdots & \hat{\underline{u}}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \hat{\underline{u}}_1^T \\ \hat{\underline{u}}_2^T \\ \vdots \\ \hat{\underline{u}}_n^T \end{bmatrix}$$

$$P^T = P^{-1}$$

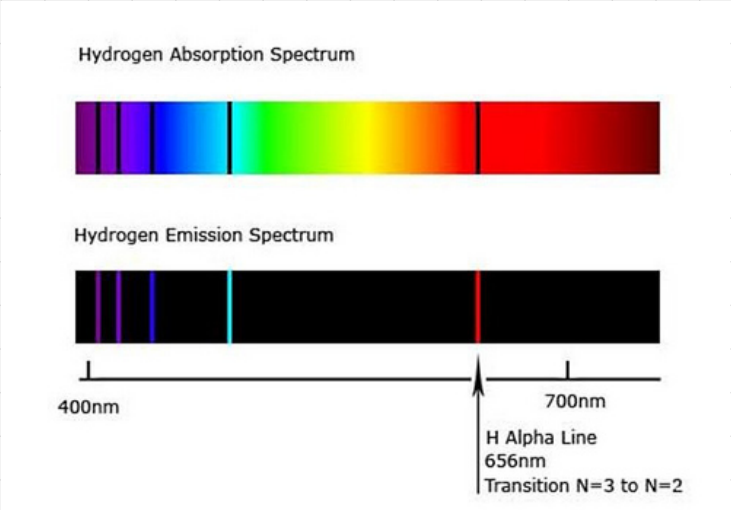
$$= \begin{bmatrix} \lambda_1 \hat{\underline{u}}_1 & \lambda_2 \hat{\underline{u}}_2 & \cdots & \lambda_n \hat{\underline{u}}_n \end{bmatrix} \begin{bmatrix} \hat{\underline{u}}_1^T \\ \hat{\underline{u}}_2^T \\ \vdots \\ \hat{\underline{u}}_n^T \end{bmatrix}$$

$$= \lambda_1 \hat{\underline{u}}_1 \hat{\underline{u}}_1^T + \lambda_2 \hat{\underline{u}}_2 \hat{\underline{u}}_2^T + \cdots + \lambda_n \hat{\underline{u}}_n \hat{\underline{u}}_n^T.$$

"Spectral Decomposition"

$$= \lambda_1 \operatorname{Proj}_{\hat{\underline{u}}_1} + \lambda_2 \operatorname{Proj}_{\hat{\underline{u}}_2} + \cdots + \lambda_n \operatorname{Proj}_{\hat{\underline{u}}_n}$$

Why "spectral"? What has any of this got to do with a spectrum?



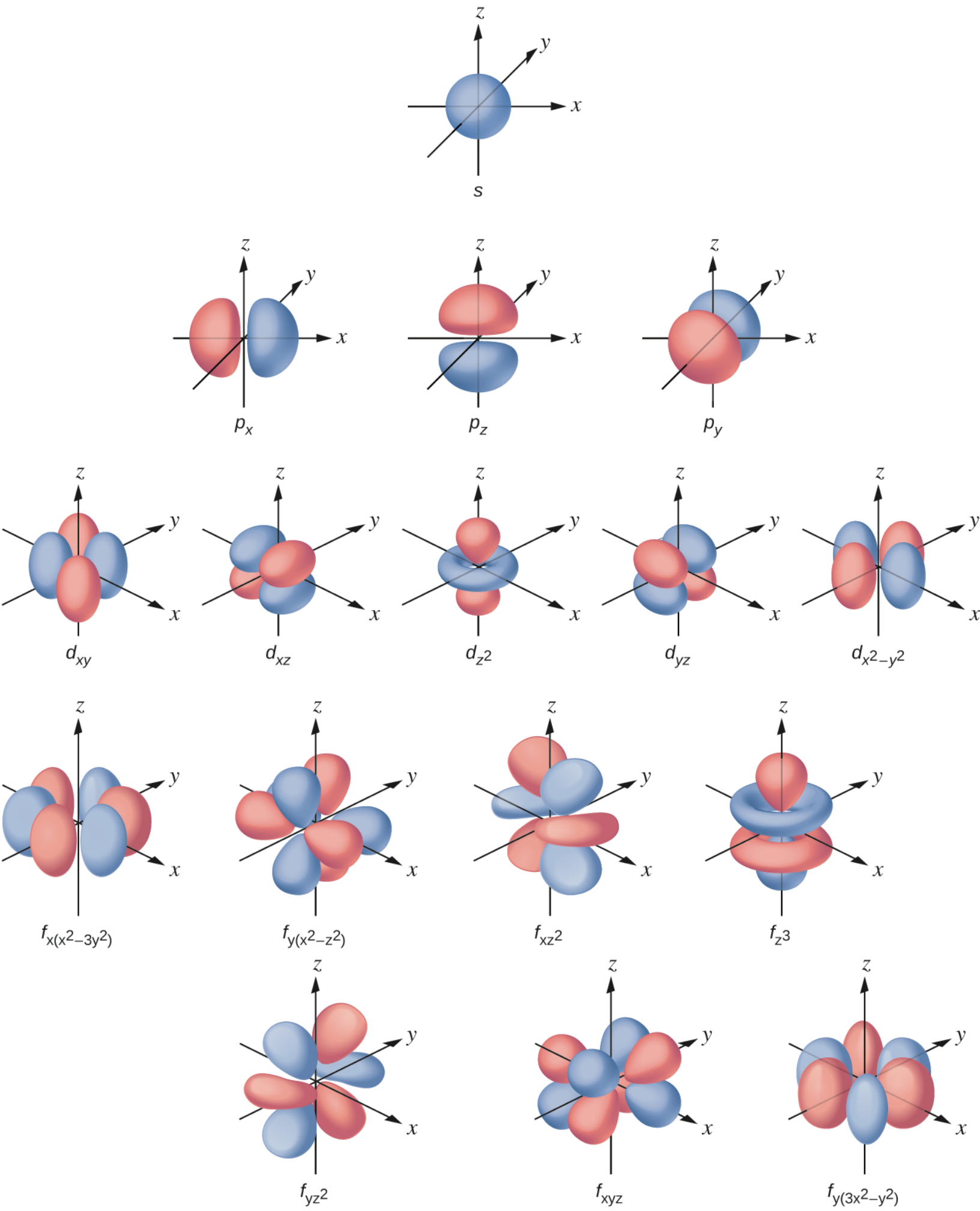
Heisenberg → "Matrix mechanics"

The vector space that describes the state of the Hydrogen atom is like $\mathbb{P}$: a space of functions.

They are "wave-functions" that represent probability distributions of how likely it is to find the particle near each point in space.

The eigenvectors are an orthonormal basis for these wave functions; they are the states with a well-defined (quantized) energy (the eigenvalues).

You've seen pictures of them before...


z
y
x
s
p_x
p_z
p_y
d_{xy}
d_{xz}
d_{z^2}
d_{yz}
$d_{x^2-y^2}$
$f_{x(x^2-3y^2)}$
$f_{y(x^2-z^2)}$
f_{xz^2}
f_{z^3}
f_{yz^2}
f_{xyz}
$f_{y(3x^2-y^2)}$