Today § 7.1 : The Spectral Theorem

Next: Review

Reminders: Please fill out your CAPEs.

MyMathLab Homework #8: Due TOMORROW by 11:59pm

FINAL EXAM: This Saturday, 11:30am – 2:30pm

GH 242, PETER 108, YORK 2722

Seating / Room Assignment on Triton Ed

Given a collection $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ of linearly independent vectors in $\mathbb{R}^n$, the Gram-Schmidt orthogonalization process produces a new collection $\{\hat{\underline{u}}_1, \hat{\underline{u}}_2, \dots, \hat{\underline{u}}_p\}$ of orthonormal vectors, with the same span as the $\underline{v}_j$'s.

$$\underline{u}_1 = \underline{v}_1 \qquad \hat{\underline{u}}_1 = \underline{u}_1 / \|\underline{u}_1\|$$

$$\underline{u}_2 = \underline{v}_2 - \frac{\underline{v}_2 \cdot \underline{u}_1}{\|\underline{u}_1\|^2} \underline{u}_1 \qquad \hat{\underline{u}}_2 = \underline{u}_2 / \|\underline{u}_2\|$$

$$\underline{u}_3 = \underline{v}_3 - \frac{\underline{v}_3 \cdot \underline{u}_1}{\|\underline{u}_1\|^2} \underline{u}_1 - \frac{\underline{v}_3 \cdot \underline{u}_2}{\|\underline{u}_2\|^2} \underline{u}_2 \qquad \hat{\underline{u}}_3 = \underline{u}_3 / \|\underline{u}_3\|$$

$$\vdots$$

The triangular pattern here can be summarized by noting this means

$$\begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_p \end{bmatrix} = \begin{bmatrix} \hat{\underline{u}}_1 & \hat{\underline{u}}_2 & \cdots & \hat{\underline{u}}_p \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ 0 & r_{22} & \cdots & r_{2p} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & r_{pp} \end{bmatrix}$$

these come up in the G-S process

Eg.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ -2 & -1 \end{bmatrix}$$

Note: $Q^TQ = I$ does not mean $Q^{-1} = Q^T$! (Only if Q is square; in general QQ^T = orthogonal projection onto $Col(Q)$.)

Why should you care about QR-factorization?

Take the square matrix case.

$$A = QR \rightsquigarrow A_1 = RQ$$

7.1 / Recall that $A \in M_{n\times n}$ is diagonalizable if there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A. ($\Leftrightarrow$ $A = PDP^{-1}$.)

What kinds of matrices have an orthonormal basis of eigenvectors?

Conclusion: if A is orthogonally diagonalizable, then

What about the converse?

Suppose A is symmetric, and happens to be diagonalizable.

$\therefore$ The eigenspaces of A span all of $\mathbb{R}^n$.

↳ we just saw that eigenvectors for distinct eigenvalues are $\perp$.

So eigenspaces for distinct eigenvalues are orthogonal.

↳ if an eigenvalue has geometric multiplicity > 1, can take any basis of this eigenspace and produce an o.n.b. ( )

<u>Conclusion</u>: There is an o.n.b. of eigenvectors of A.

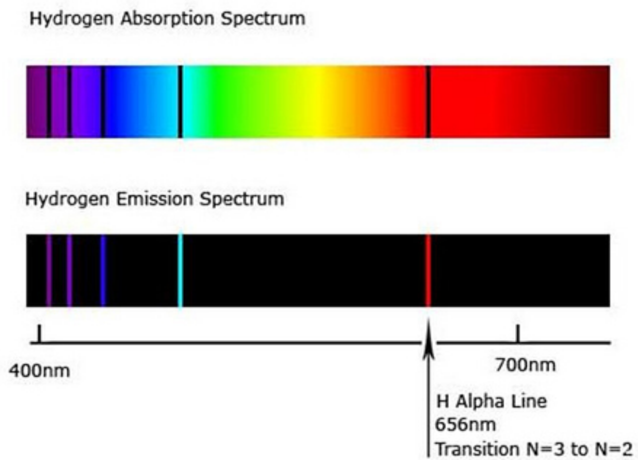
I.e. A is orthogonally diagonalizable.

The Spectral Theorem

Every symmetric matrix is orthogonally diagonalizable.

This gives the spectral decomposition of a symmetric matrix:

Why "spectral"? What has any of this got to do with a spectrum?

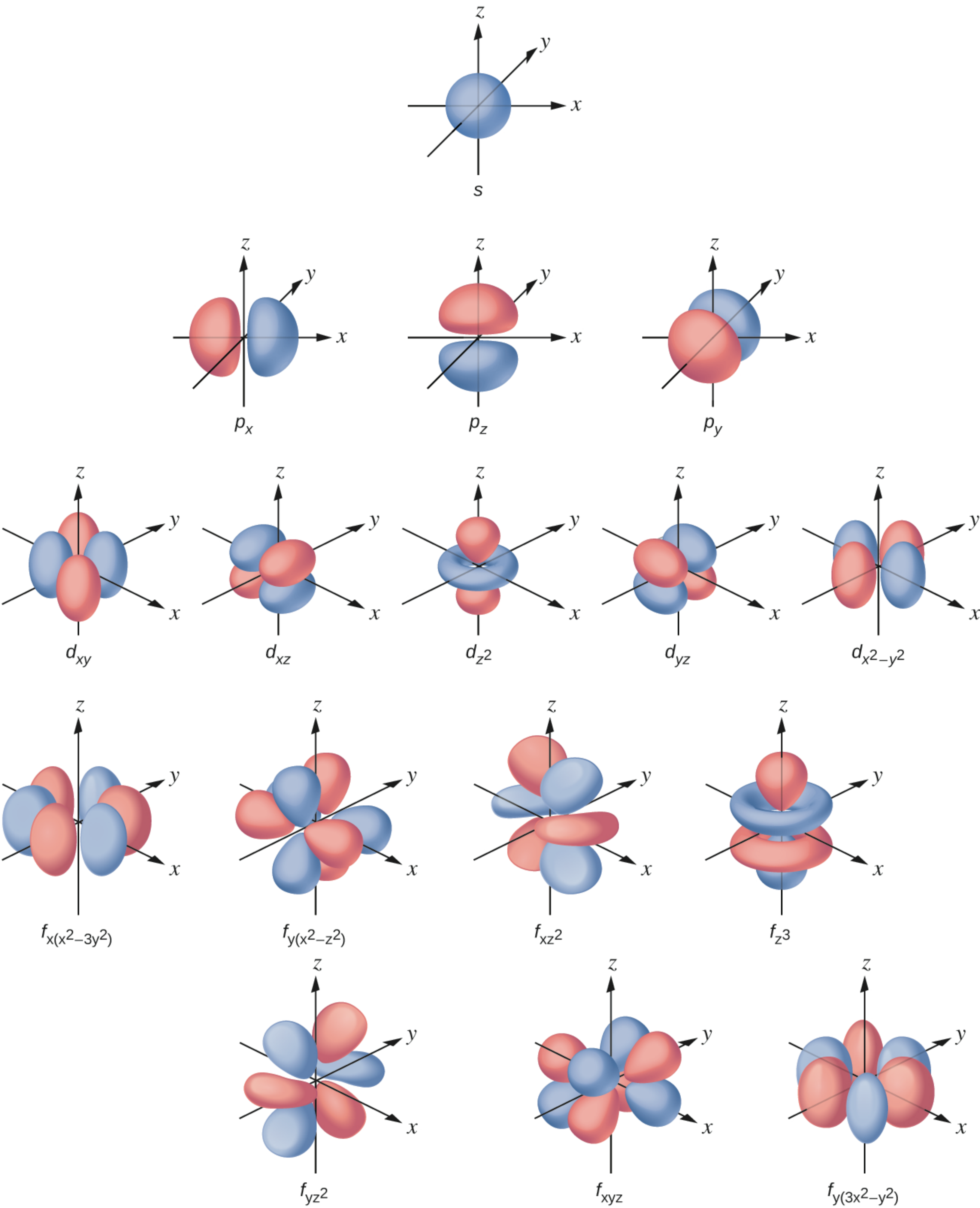


The vector space that describes the state of the Hydrogen atom is like $\mathbb{P}$: a space of functions.

They are "wave-functions" that represent probability distributions of how likely it is to find the particle near each point in space.

The eigenvectors are an orthonormal basis for these wave functions; they are the states with a well-defined (quantized) energy (the eigenvalues).

You've seen pictures of them before...



z
y
x
s
p_x
p_z
p_y
d_{xy}
d_{xz}
d_{z^2}
d_{yz}
$d_{x^2-y^2}$
$f_{x(x^2-3y^2)}$
$f_{y(x^2-z^2)}$
f_{xz^2}
f_{z^3}
f_{yz^2}
f_{xyz}
$f_{y(3x^2-y^2)}$